Math 309 - Worksheet - Linear algebra review

$$A = \begin{bmatrix} 2 & -1 \\ -1 & 2 \end{bmatrix}, B = \begin{bmatrix} i & 1 \\ 1 & -i \end{bmatrix}, C = \begin{bmatrix} 3 & i \\ -i & 1 \end{bmatrix}, F = \begin{bmatrix} 1 & -2 & 3 \\ -1 & 1 & -2 \\ 2 & -1 & 3 \end{bmatrix}, v = \begin{bmatrix} 1 \\ 2 \end{bmatrix}.$$

1. **Determinant:** Compute $\det B, \det C, \det F$.

Formulas: $\det\begin{bmatrix} a & b \\ c & d \end{bmatrix} = ad - bc$

$$\det\begin{pmatrix} a & b & c \\ d & e & f \\ g & h & \ell \end{pmatrix} = a\det\begin{pmatrix} e & f \\ h & \ell \end{pmatrix} - b\det\begin{pmatrix} d & f \\ g & \ell \end{pmatrix} + c\det\begin{pmatrix} d & e \\ g & h \end{pmatrix}$$

$\det B = i(-i) - 1 = 0$

$\det C = 3 - (-i)i = 2$

$\det F = 1(3-2) + 2(-3+4) + 3(1-2) = 1 + 2 - 3 = 0$

Fact:

$\det M = 0 \iff$ rows are linearly dependent

$\iff$ columns are linearly dependent

$\iff$ M not invertible

For 2×2 matrices

$\det\begin{bmatrix} a & b \\ c & d \end{bmatrix} = 0 \iff [a \quad b] = r[c \quad d] = [rc \quad rd]$ for some $r \in \mathbb{C}$

2. **Linear systems with invertible matrix:**

Consider the equation $Cx = v$ where $x = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$.

$$\begin{bmatrix} 3 & i \\ -i & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \end{bmatrix} \iff \begin{cases} 3x_1 + ix_2 = 1 & (1) \\ -ix_1 + x_2 = 2 & (2) \end{cases}$$

Find the solution to $Cx = v$.

$$(2) \Rightarrow x_2 = 2 + ix_1$$

$$(1) \Rightarrow 3x_1 + i(2 + ix_1) = 1$$

$$2x_1 = 1 - 2i$$

$$\begin{cases} x_1 = \frac{1}{2} - i \\ x_2 = 2 + ix_1 = 2 + i(\frac{1}{2} - i) = 3 + \frac{i}{2} \end{cases}$$

<u>Method 2</u>: $Cx = v$

Formula: $\begin{bmatrix} a & b \\ c & d \end{bmatrix}^{-1} = \frac{1}{ad-bc} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$

$\det C \neq 0$ so C^{-1} exists

$$\Rightarrow \underbrace{C^{-1}C}_{I} x = C^{-1}v$$

$$\Rightarrow x = C^{-1}v = \frac{1}{2}\begin{bmatrix} 1 & -i \\ i & 3 \end{bmatrix} \begin{bmatrix} 1 \\ 2 \end{bmatrix} = \begin{bmatrix} \frac{1}{2} - i \\ \frac{i}{2} + 3 \end{bmatrix}$$

Find the solution to $Cx = 0$ (homogeneous equation).

$$\Rightarrow x = C^{-1}0 = 0, \text{ i.e. } \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \text{ trivial solution}$$

Fact: $\det C \neq 0$

$\Rightarrow Cx = v$ has a unique solution $x = C^{-1}v$.

$\Rightarrow Cx = 0$ " " " $x = 0$.

3. **Linear systems with noninvertible matrix:**

Find the set of solutions to $Bx = 0$.

$$\begin{bmatrix} i & 1 \\ 1 & -i \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \iff \begin{cases} ix_1 + x_2 = 0 & (1) \\ x_1 - ix_2 = 0 & (2) \end{cases}$$

$$\iff \begin{cases} ix_1 + x_2 = 0 \\ i(2) - (1):\ 0 + 0 = 0 \end{cases}$$

Redundant equations due to $\det B = 0$

$$\iff ix_1 + x_2 = 0, \text{ i.e. } x_2 = -ix_1$$

set of soln: $\left\{ \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} x_1 \\ -ix_1 \end{bmatrix} = \begin{bmatrix} 1 \\ -i \end{bmatrix} x_1,\ x_1 \in \mathbb{R} \right\}$ has dim $= 1$

Find the set of solutions to $Bx = v$.

$$\begin{bmatrix} i & 1 \\ 1 & -i \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \end{bmatrix} \iff \begin{cases} ix_1 + x_2 = 1 & (1) \\ x_1 - ix_2 = 2 & (2) \end{cases}$$

$$\iff \begin{cases} ix_1 + x_2 = 1 \\ i(2) - (1)\ \ 0 = 2i - 1 \end{cases}$$

contradiction

$\underbrace{\text{no solution}}$

Find the set of solutions to $Bx = \begin{bmatrix} 1 \\ -i \end{bmatrix}$.

$$\begin{bmatrix} i & 1 \\ 1 & -i \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 1 \\ -i \end{bmatrix} \iff \begin{cases} ix_1 + x_2 = 1 & (1) \\ x_1 - ix_2 = -i & (2) \end{cases}$$

$$\iff \begin{cases} ix_1 + x_2 = 1 \\ i(2) - (1):\ 0 = 0 \end{cases}$$

$$\iff ix_1 + x_2 = 1, \text{ i.e. } x_2 = 1 - ix_1$$

next page

Set of soln:

$$\left\{ \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} x_1 \\ 1-i x_1 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \end{bmatrix} + \underbrace{\begin{bmatrix} 1 \\ -i \end{bmatrix} x_1}_{\text{Soln to } Bx=0}, \ x_1 \in \mathbb{R} \right\}$$ has dim=1

Fact: $\det B = 0$

$\Rightarrow$ $Bx = 0$ has infinitely many solutions

$\Rightarrow$ $Bx = v$ either has no solution or has infinitely many solutions

4. **Linear systems with noninvertible matrix cont'd:**

Find the set of solutions to $Fx = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$.

$$\begin{bmatrix} 1 & -2 & 3 \\ -2 & 1 & -2 \\ 2 & -1 & 3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \quad \rightsquigarrow \quad \begin{cases} x_1 - 2x_2 + 3x_3 = 0 & ① \\ -x_1 + x_2 - 2x_3 = 0 & ② \\ 2x_1 - x_2 + 3x_3 = 0 & ③ \end{cases}$$

$$① + ② \Rightarrow \quad -x_2 + x_3 = 0$$
$$x_2 = x_3$$

$$② \Rightarrow \quad -x_1 + x_3 - 2x_3 = 0$$
$$x_1 = -x_3$$

$$③ \Rightarrow \quad -2x_3 - x_3 + 3x_3 = 0$$
$$0 = 0 \quad , \; x_3 \text{ arbitrary}$$

Solution set

$$\left\{ \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} -1 \\ 1 \\ 1 \end{bmatrix} x_3 , \quad x_3 \in \mathbb{R} \right\}$$

Alternatively, more systematically, can do

$$① \Rightarrow x_1 = 2x_2 - 3x_3$$

$$② \Rightarrow -(2x_2 - 3x_3) + x_2 - 2x_3 = 0$$
$$\Rightarrow \quad x_2 = x_3 \quad , \quad \text{so} \quad x_1 = 2x_2 - 3x_3 = -x_3$$

$$③ \Rightarrow -2x_3 - x_3 + 3x_3 = 0 , \quad x_3 \text{ arbitrary}$$

5. **Linear (in)dependence:**
Are the three column vectors of F linearly independent? If they are linearly dependent, find a linear relation among them.

Column vectors are dependent since $\det F = 0$.

So want to find c_1, c_2, c_3, not all 0, s.t.

$$c_1 \begin{bmatrix} 1 \\ -1 \\ 2 \end{bmatrix} + c_2 \begin{bmatrix} -2 \\ 1 \\ -1 \end{bmatrix} + c_3 \begin{bmatrix} 3 \\ -2 \\ 3 \end{bmatrix} = 0$$

i.e.

$$\begin{bmatrix} 1 & -2 & 3 \\ -1 & 1 & -2 \\ 2 & -1 & 3 \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

We already found answers to this in Problem 4 that

$$\begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix} = \begin{bmatrix} -1 \\ 1 \\ 1 \end{bmatrix} c_3,$$ where any $c_3 \in \mathbb{R}$ will do.

Pick $c_3 = 1$

$$-\begin{bmatrix} 1 \\ -1 \\ 2 \end{bmatrix} + \begin{bmatrix} -2 \\ 1 \\ -1 \end{bmatrix} + \begin{bmatrix} 3 \\ -2 \\ 3 \end{bmatrix} = 0$$

Note about linear independence.

Vectors $\vec{v}_1, \ldots, \vec{v}_n$ are linearly independent if

$$c_1\vec{v}_1 + c_2\vec{v}_2 + \cdots + c_n\vec{v}_n = 0$$
$$\Rightarrow \quad c_1 = c_2 = \cdots = c_n = 0$$

<u>Note</u> : If $c_1\vec{v}_1 + c_2\vec{v}_2 + c_3\vec{v}_3 = 0$ but at least one of the c's is not zero, say $c_2 \neq 0$. Then

$$c_2\vec{v}_2 = -c_1\vec{v}_1 - c_3\vec{v}_3$$
$$\vec{v}_2 = -\frac{c_1}{c_2}\vec{v}_1 - \frac{c_3}{c_2}\vec{v}_3$$

(can divide by c_2 because $c_2 \neq 0$)

i.e. $\vec{v}_2$ is a linear combination of $\vec{v}_1$ and $\vec{v}_3$.

So, $\vec{v}_1, \vec{v}_2, \vec{v}_3$ dependent

<u>Example:</u> Suppose $\vec{v}_1, \vec{v}_2, \vec{v}_3, \vec{v}_4$ are linearly independent and

$$c_1\vec{v}_1 + c_2\vec{v}_2 + c_3\vec{v}_3 + c_4\vec{v}_4 = 7\vec{v}_2$$

What are c_1, c_2, c_3, c_4?

Answer : $c_1 = c_3 = c_4 = 0, \quad c_2 = 7$ (just match the coefficients)

This is because

$$c_1\vec{v}_1 + (c_2 - 7)\vec{v}_2 + c_3\vec{v}_3 + c_4\vec{v}_4 = 0$$

linear independence $\Rightarrow$ $c_1 = c_2 - 7 = c_3 = c_4 = 0$

6. **Eigenvalues and eigenvectors:**

Given a matrix M, if $Mx = \lambda x$, equivalently $(M - \lambda I)x = 0$, for some $x \neq 0$, then λ is an **eigenvalue** of M and x is an **eigenvector** of M corresponding to λ.

Note that there exists $x \neq 0$ such that $(M - \lambda I)x = 0$ if and only if $\det(M - \lambda I) = 0$.

$A = \begin{bmatrix} 2 & -1 \\ -1 & 2 \end{bmatrix}$

(a) Find all the eigenvalues of the matrix A and the set of eigenvectors corresponding to each eigenvalue.

(b) Write $A = PDP^{-1}$ where D is a diagonal matrix with the diagonal entries being the eigenvalues.

Find eigenvalue :

$$0 = \det(A - \lambda I) = \det\begin{pmatrix} -2-\lambda & -1 \\ -1 & 2-\lambda \end{pmatrix} = (2-\lambda)^2 - 1 = \lambda^2 - 4\lambda + 3 = (\lambda - 1)(\lambda - 3)$$

$\boxed{\lambda_1 = 1}$, $\boxed{\lambda_2 = 3}$

Eigenvector corresp. to $\lambda_1 = 1$

Solve $(A - \lambda_1 I)\, v = 0$

$$\begin{bmatrix} 2-1 & -1 \\ -1 & 2-1 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \iff v_1 - v_2 = 0 \text{ , i.e. } v_1 = v_2$$

set of eigenvectors corresponding to $\lambda_1 = 1$:

$$\left\{ \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \end{bmatrix} v_1 , \quad v_1 \neq 0 \right\}$$

Eigenvector corresp. to $\lambda_2 = 3$

Solve $(A - \lambda_2 I)\, w = 0$

$$\begin{bmatrix} 2-3 & -1 \\ -1 & 2-3 \end{bmatrix} \begin{bmatrix} w_1 \\ w_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \iff -w_1 - w_2 = 0 \text{ , i.e. } w_1 = -w_2$$

set of eigenvectors corresponding to $\lambda_2 = 3$:

$$\left\{ \begin{bmatrix} w_1 \\ w_2 \end{bmatrix} = \begin{bmatrix} -1 \\ 1 \end{bmatrix} w_2 , \quad w_2 \neq 0 \right\}$$

(b) $P = \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix}$, $P^{-1} = \begin{bmatrix} \frac{1}{2} & \frac{1}{2} \\ -\frac{1}{2} & \frac{1}{2} \end{bmatrix}$, $A = P \underbrace{\begin{bmatrix} 1 & 0 \\ 0 & 3 \end{bmatrix}}_{= D} P^{-1}$

eigenvec for λ_1 eigenvec for λ_2

A note about complex numbers :

* $i^2 = -1$

* $\frac{1}{3+4i} = \frac{3-4i}{(3+4i)(3-4i)} = \frac{3-4i}{3^2+4^2}$

$$= \frac{1}{25}(3-4i)$$

* $\lambda = a+bi$, $\bar{\lambda} = a-bi$

if M is real,

then { if v is an eigenvector with eigenvalue λ, i.e. $Mv = \lambda v$

then $\boxed{M\bar{v}} = \bar{M}\bar{v} = \overline{Mv} = \overline{\lambda v} = \boxed{\bar{\lambda}\,\bar{v}}$

i.e. $\bar{v}$ is an eigenvector with eigenvalue $\bar{\lambda}$