Math 309 Homework 7
(6 problems)

1. Consider the system of 2nd order equations

$$\begin{cases} x'' = ny \\ y'' = -nx \end{cases},$$

where $n \geq 1$ is some constant. Now we write this as an equivalent system of 1st order equations

$$\begin{cases} x_1' = x_2 \\ x_2' = ny_1 \\ y_1' = y_2 \\ y_2' = -nx_1 \end{cases}, \quad \text{i.e.} \quad \begin{bmatrix} x_1' \\ x_2' \\ y_1' \\ y_2' \end{bmatrix} = A \begin{bmatrix} x_1 \\ x_2 \\ y_1 \\ y_2 \end{bmatrix}, \quad \text{where } A = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & n & 0 \\ 0 & 0 & 0 & 1 \\ -n & 0 & 0 & 0 \end{bmatrix}.$$

Find the general solution to the above system of first order equations in terms of real valued functions.

Hint: here I explain and provide the eigenvalues and eigenvectors of the matrix A. Then you can just use the eigenvalues and eigenvectors that I boxed below without explanation. To find the eigenvalues, we need to find λ such that

$$\begin{aligned} & \det(A - \lambda I) = \lambda^4 + n^2 = 0 \\ \Rightarrow\ & \lambda^4 = -n^2 \\ \Rightarrow\ & \lambda = (-1)^{\frac{1}{4}}\sqrt{n}. \end{aligned}$$

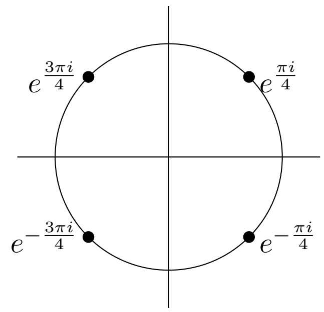
Note that $-1 = e^{ik\pi} = \cos(k\pi) + i\sin(k\pi)$, where k is an odd integer. So the 4th roots of -1 are

$$(-1)^{\frac{1}{4}} = e^{\frac{ik\pi}{4}} = \cos\frac{k\pi}{4} + i\sin\frac{k\pi}{4}.$$

From this it appears like there are infinitely many 4th roots of -1, one for each odd integer k; however, most of these are repetitions since cosine and sine are 2π periodic. Distinct roots can be represented by the k's where $\frac{k\pi}{4}$ are in a 2π interval such as $(-\pi, \pi]$, which are $k = \pm1, \pm3$ that correspond to $\frac{k\pi}{4} = \pm\frac{\pi}{4}, \pm\frac{3\pi}{4}$. So there are four 4th roots of -1, which are given by

$$\begin{aligned} e^{\frac{\pi i}{4}} &= \cos\frac{\pi}{4} + i\sin\frac{\pi}{4} = \frac{1}{\sqrt{2}} + i\frac{1}{\sqrt{2}}, \\ e^{-\frac{\pi i}{4}} &= \cos\left(-\frac{\pi}{4}\right) + i\sin\left(-\frac{\pi}{4}\right) = \frac{1}{\sqrt{2}} - i\frac{1}{\sqrt{2}}, \\ e^{\frac{3\pi i}{4}} &= \cos\frac{3\pi}{4} + i\sin\frac{3\pi}{4} = -\frac{1}{\sqrt{2}} + i\frac{1}{\sqrt{2}}, \\ e^{-\frac{3\pi i}{4}} &= \cos\left(-\frac{3\pi}{4}\right) + i\sin\left(-\frac{3\pi}{4}\right) = -\frac{1}{\sqrt{2}} - i\frac{1}{\sqrt{2}}. \end{aligned}$$

Below is a plot of these 4 points on the complex plane



In summary, the eigenvalues of A are

$$\boxed{\sqrt{\frac{n}{2}} \pm i\sqrt{\frac{n}{2}} \quad \text{and} \quad -\sqrt{\frac{n}{2}} \pm i\sqrt{\frac{n}{2}}}$$

and notice that they come in conjugate pairs which is a consequence of A being a real matrix.

For the conjugate pair $\sqrt{\frac{n}{2}} \pm i\sqrt{\frac{n}{2}}$, we can find eigenvectors v corresponding to $\sqrt{\frac{n}{2}} + i\sqrt{\frac{n}{2}}$ by solving

$$\left[A - \left(\sqrt{\frac{n}{2}} + i\sqrt{\frac{n}{2}}\right) I\right] v = \begin{bmatrix} -\sqrt{\frac{n}{2}} - i\sqrt{\frac{n}{2}} & 1 & 0 & 0 \\ 0 & -\sqrt{\frac{n}{2}} - i\sqrt{\frac{n}{2}} & n & 0 \\ 0 & 0 & -\sqrt{\frac{n}{2}} - i\sqrt{\frac{n}{2}} & 1 \\ -n & 0 & 0 & -\sqrt{\frac{n}{2}} - i\sqrt{\frac{n}{2}} \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ v_3 \\ v_4 \end{bmatrix} = 0,$$

i.e.

$$\begin{aligned}
&① \quad \left(-\sqrt{\tfrac{n}{2}} - i\sqrt{\tfrac{n}{2}}\right) v_1 + v_2 = 0 && \Rightarrow \quad v_2 = \left(\sqrt{\frac{n}{2}} + i\sqrt{\frac{n}{2}}\right) v_1, \\
&② \quad \left(-\sqrt{\tfrac{n}{2}} - i\sqrt{\tfrac{n}{2}}\right) v_2 + nv_3 = 0 && \Rightarrow \quad v_3 = \frac{1}{n}\left(\sqrt{\frac{n}{2}} + i\sqrt{\frac{n}{2}}\right) v_2, \\
&③ \quad \left(-\sqrt{\tfrac{n}{2}} - i\sqrt{\tfrac{n}{2}}\right) v_3 + v_4 = 0 && \Rightarrow \quad v_4 = \left(\sqrt{\frac{n}{2}} + i\sqrt{\frac{n}{2}}\right) v_3, \\
&④ \quad -nv_1 + \left(-\sqrt{\tfrac{n}{2}} - i\sqrt{\tfrac{n}{2}}\right) v_4 = 0 && \Rightarrow \quad v_1 = -\frac{1}{n}\left(\sqrt{\frac{n}{2}} + i\sqrt{\frac{n}{2}}\right) v_4.
\end{aligned}$$

We can take advantage of the fact that $\frac{1}{\sqrt{2}} + i\frac{1}{\sqrt{2}} = e^{\frac{i\pi}{4}}$ to make multiplication easier, e.g. $\left(\frac{1}{\sqrt{2}} + i\frac{1}{\sqrt{2}}\right)^2 = \left(e^{\frac{i\pi}{4}}\right)^2 = e^{\frac{i\pi}{2}} = i$. We then get that

$$\begin{aligned}
&② \Rightarrow \quad v_3 = \frac{1}{\sqrt{n}} e^{\frac{i\pi}{4}} v_2 = e^{\frac{i\pi}{2}} v_1 = iv_1, \\
&③ \Rightarrow \quad v_4 = \left(\sqrt{\frac{n}{2}} + i\sqrt{\frac{n}{2}}\right) v_3 = i\left(\sqrt{\frac{n}{2}} + i\sqrt{\frac{n}{2}}\right) v_1 = \left(-\sqrt{\frac{n}{2}} + i\sqrt{\frac{n}{2}}\right) v_1
\end{aligned}$$

So

$$v = \begin{bmatrix} v_1 \\ v_2 \\ v_3 \\ v_4 \end{bmatrix} = \begin{bmatrix} 1 \\ \sqrt{\frac{n}{2}} + i\sqrt{\frac{n}{2}} \\ i \\ -\sqrt{\frac{n}{2}} + i\sqrt{\frac{n}{2}} \end{bmatrix} v_1, \quad v_1 \neq 0 \text{ arbitrary.}$$

So

$$\boxed{\text{an eigenvector corresponding to } \sqrt{\frac{n}{2}}+i\sqrt{\frac{n}{2}} \text{ is } \begin{bmatrix} 1 \\ \sqrt{\frac{n}{2}}+i\sqrt{\frac{n}{2}} \\ i \\ -\sqrt{\frac{n}{2}}+i\sqrt{\frac{n}{2}} \end{bmatrix}}.$$

For the conjugate pair $-\sqrt{\frac{n}{2}} \pm i\sqrt{\frac{n}{2}}$, we can find the eigenvectors corresponding to $-\sqrt{\frac{n}{2}}+i\sqrt{\frac{n}{2}}$ via a similar calculation and get that

$$\boxed{\text{an eigenvector corresponding to } -\sqrt{\frac{n}{2}}+i\sqrt{\frac{n}{2}} \text{ is } \begin{bmatrix} 1 \\ -\sqrt{\frac{n}{2}}+i\sqrt{\frac{n}{2}} \\ -i \\ \sqrt{\frac{n}{2}}+i\sqrt{\frac{n}{2}} \end{bmatrix}}.$$

2. Let $f(x) = 1$ with $0 \leq x \leq \pi$.
   (a) Find the Fourier cosine series for $f(x)$.
   (b) Find the Fourier sine series for $f(x)$.

3. (a) Solve the given boundary value problem or else show that it has no solution

$$y'' + y = 0, \quad y(0) = 0, \quad y'(\pi) = 1.$$

   (b) Solve the given boundary value problem or else show that it has no solution

$$y'' + y = 0, \quad y'(0) = 1, \quad y(L) = 0.$$

4. Find the eigenvalues and eigenfunctions of the given boundary value problem.

$$y'' + \lambda y = 0, \quad y(0) = 0, \quad y'(\pi) = 0.$$

5. (a) Determine whether the method of separation of variables can be used to replace the given partial differential equation by a pair of ordinary differential equations. If so, find the equations.

$$u_{xx} + (x+y)u_{yy} = 0.$$

   (b) Determine whether the method of separation of variables can be used to replace the given partial differential equation by a pair of ordinary differential equations. If so, find the equations.

$$u_{xx} + u_{yy} + xu = 0.$$

6. Given a Hamiltonian function $H(x,p)$, the Hamilton-Jacobi equation is

$$\frac{\partial W(x,t)}{\partial t} = -H\left(x, \frac{\partial W(x,t)}{\partial x}\right).$$

So for $H(x,p) = \frac{p^2}{2} + V(x)$, the Hamilton-Jacobi equation for $W(x,t)$ is

$$\frac{\partial W}{\partial t} = -\frac{1}{2}\left(\frac{\partial W}{\partial x}\right)^2 - V(x).$$

(a) Determine whether the method of separation of variables can be used to replace the above Hamilton-Jacobi equation for $W(x,t) = f(x)g(t)$ by a pair of ordinary differential equations, one for $f(x)$ and one for $g(t)$. If so, find the equations.

(b) Now let us look for solutions of the form $W(x,t) = h(x) + r(t)$. Find $r(t)$.