Math 309 Homework 3
(6 problems)

1. For simplicity, in this problem you can assume $x(t)$ is a scalar valued function, i.e. not vector valued, (though everything here works in exactly the same way even if $x(t)$ is a vector valued function).

   (a) Show that if $x^{(1)}(t)$ and $x^{(2)}(t)$ are solutions to a homogeneous linear first order equation, i.e. an equation of the form $x' = p(t)x$, then $x(t) = c_1 x^{(1)}(t) + c_2 x^{(2)}(t)$ is also a solution to $x' = p(t)x$.

   (b) Now suppose $x^{(1)}(t)$ and $x^{(2)}(t)$ are nonzero solutions to the following homogeneous, but nonlinear, first order equation,
   $$x' = x^2.$$
   Show that $x = x^{(1)}(t) + x^{(2)}(t)$ is not a solution to the above equation $x' = x^2$.

   (c) Now suppose $x^{(1)}(t)$ and $x^{(2)}(t)$ are solutions to the following linear, but nonhomogeneous, first order equation,
   $$x' = x + 2.$$
   Show that $x = x^{(1)}(t) + x^{(2)}(t)$ is not a solution to the above equation $x' = x + 2$. This x is actually a solution to a different equation, and what is that equation?

2. (Continuation of HW2 #3) In HW2, #3, we solved the below system of equations,
   $$\begin{array}{rcl} x_1' &=& x_1 - 2x_2 \\ x_2' &=& 3x_1 - 4x_2 \end{array},$$
   by writing this system as a single 2nd order equation and solving that 2nd order equation first.

   (a) Now forget about writing the system as a 2nd order equation. Just find the general solution to the above first order system $x' = Ax$ directly by finding the eigenvalues and eigenvectors fo the coefficient matrix A.

   (b) Describe the behaviors of the solution as $t \to \infty$ and as $t \to -\infty$, (i.e. are solutions $x(t)$ going to 0 or ∞?)

   (c) Draw a few trajectories of solutions in the x_1-x_2 plane. Draw the trajectories parallel to the eigenvectors and also include a couple of trajectories that are not parallel to the the eigenvectors. Indicate the direction of flow by drawing an arrow on each trajectory.

3. (Continuation of HW3 #2)

   (a) Find the solution to the above system of equations given the following initial condition
   $$x(0) = \begin{bmatrix} -1 \\ -2 \end{bmatrix}$$

(b) Plot the point $x(0)$ on the x_1-x_2 plane.

(c) Draw the trajectory of the solution to part (a) on the x_1-x_2 plane for $t \in [0, \infty)$. Indicate the direction of flow.

(d) Plot the point $x(1)$ on your drawing in part (c).

4. (Rolling down a potential hill, a continuation of HW2 #6) In HW2, #6b, we wrote down the following system describing the motion of the particle

$$\begin{bmatrix} x' \\ p' \end{bmatrix} = A \begin{bmatrix} x \\ p \end{bmatrix}, \qquad \text{where } A = \begin{bmatrix} 0 & 1/m \\ 2/5 & 0 \end{bmatrix}.$$

(a) Find the general solution, $\begin{bmatrix} x(t) \\ p(t) \end{bmatrix}$, to this system by finding the eigenvalues and eigenvectors of A. (The answer you get in the end should be the same as that in HW2 #6a.)

(b) Suppose $x(0) = 0$ and $p(0) = 0$. Find the solution $\begin{bmatrix} x(t) \\ p(t) \end{bmatrix}$ subject to this initial condition.

(c) Again $x(0) = 0$ and $p(0) = m$ (i.e. initial velocity points to the right). Find the solution $\begin{bmatrix} x(t) \\ p(t) \end{bmatrix}$ subject to this initial condition.

(d) What is the behavior of the solution in part (c) as $t \to \infty$? (That is, are solutions going to 0 or ∞?)

(e) Let $m = 1$. Draw the trajectory of solution in part (c) for for $t \in [0, \infty)$ in the x-p plane and remember to indicate the direction of flow.

5. (Pendulum, a continuation of HW1 #8) In HW1 #8, we discussed that the equation of motion for a frictionless pendulum of bob mass m and rod length L is

$$\theta''(t) = -\frac{g}{L}\sin\theta.$$

This is a nonlinear equation, and we linearized it in HW1 by considering very small $\theta \approx 0$, in which case $\sin\theta \approx \theta$.

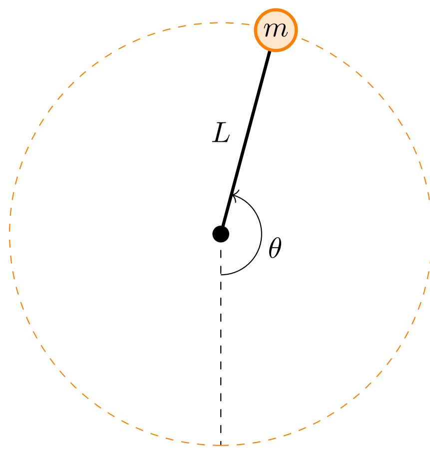
Now let's linearize it near a different point by considering $\theta \approx \pi$, so the bob of the pendulum is near the very top as pictured below. In this case, by considering a Taylor expansion of $\sin\theta$ around π, we get that $\sin\theta \approx -(\theta - \pi)$. Then we get that for $\theta \approx \pi$, we can approximate the above nonlinear equation by the following linearized equation

$$\theta''(t) = \frac{g}{L}(\theta - \pi).$$

Now let $\tilde{\theta} = \theta - \pi$, so when $\theta = \pi$, we have $\tilde{\theta} = 0$. Then the above equation becomes

$$\tilde{\theta}'' = \frac{g}{L}\tilde{\theta}.$$

For the rest of this problem, we only consider this very last linearized equation in $\tilde{\theta}$.



(a) Denote by $\omega(t) = \frac{d\theta(t)}{dt}$. Write the above linearized equation in $\tilde{\theta}$ as a system of first order equations

$$\begin{bmatrix} \tilde{\theta}' \\ \omega' \end{bmatrix} = A \begin{bmatrix} \tilde{\theta} \\ \omega \end{bmatrix}.$$

(b) Find the general solution to the system in part (a) by finding the eigenvalues and eigenvectors of A.

(c) Now set $\frac{g}{L} = 1$. Draw a few trajectories of solutions in the $\tilde{\theta}$-ω plane. Draw the trajectories parallel to the eigenvectors and also include a couple of trajectories that are not parallel to the the eigenvectors. Indicate the direction of flow by drawing an arrow on each trajectory.

(d) Again keep $\frac{g}{L} = 1$. Draw the trajectory determined by the initial condition $\tilde{\theta}(0) = 0$ and $\omega(0) = 1$ for $t \in [0, \infty)$ in the $\tilde{\theta}$-ω plane. Then on the same plane, also draw the trajectory determined by the initial condition $\tilde{\theta}(0) = 0.3$ and $\omega(0) = 0$ for $t \in [0, \infty)$

6. Consider the following system

$$x' = \begin{bmatrix} 2 & -1 \\ 3 & -2 \end{bmatrix} x.$$

(a) Find the general solution by finding the eigenvalues and eigenvectors of the coefficient matrix.

(b) Draw a few trajectories of solutions in the x_1-x_2 plane. Draw the trajectories parallel to the eigenvectors and also include a couple of trajectories that are not parallel to the the eigenvectors. Indicate the direction of flow by drawing an arrow on each trajectory.

(c) Draw the trajectory determined by the initial condition $x_1(0) = 2$ and $x_2(0) = 1$ for $t \in [0, \infty)$ in the x_1-x_2 plane.