§7.8 cont'd

Ex2 $x' = Ax$, $A = \begin{bmatrix} 1 & 1 & 1 \\ 2 & 1 & -1 \\ -3 & 2 & 4 \end{bmatrix}$

Eigenval: $\lambda = 2, 2, 2$ $\qquad (\lambda-2)^3 = 0$

Eigenvec: $v = \begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix}$

Generalized eigenvec:

* $(A-2I)u = v$ $\qquad$ (so $(A-2I)^2 u = (A-2I)v = 0$)

$u = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$

* $(A-2I)w = u$ $\qquad$ (so $(A-2I)^3 w = (A-2I)^2 u = 0$)
$(A-2I)^2 w = (A-2I)u = v$

$w = \begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix}$

$$e^{At} = e^{2t} e^{(A-2I)t}$$

$$= e^{2t} \left[I + (A-2I)t + \frac{(A-2I)^2 t^2}{2} \right]$$

$$e^{At} v = e^{2t} v$$

$$e^{At} u = e^{2t} u + te^{2t} v$$

$$e^{At} w = e^{2t} w + te^{2t} u + \frac{t^2 e^{2t}}{2} v$$

$$x(t) = c_1 e^{2t} v + c_2 \left(e^{2t} u + te^{2t} v \right) + c_3 \left(e^{2t} w + te^{2t} u + \frac{t^2 e^{2t}}{2} v \right)$$

$$A = P \underbrace{\begin{bmatrix} 2 & 1 & 0 \\ 0 & 2 & 1 \\ 0 & 0 & 2 \end{bmatrix}}_{\text{Jordan form}} P^{-1}, \qquad P = \begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & 0 \\ -1 & 1 & 2 \end{bmatrix}$$

Ex3 $\quad A = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 3 & 0 \\ 0 & 1 & 1 \end{bmatrix}$

$$0 = \det(A - \lambda I) = -(\lambda-1)^2(\lambda-3)$$

$$\lambda = 3,\ 1,\ 1$$

eigenvec for 3 : $\quad u = \begin{bmatrix} 0 \\ 2 \\ 1 \end{bmatrix}$

eigenvec for 1 : $\quad v = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$

gen. eigenvec for 1: $\quad (A - I)\,w = v$

$$w = \begin{bmatrix} -2 \\ 1 \\ 0 \end{bmatrix}$$

$$A = P \begin{bmatrix} \boxed{3} & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix} P^{-1} \quad , \quad P = \begin{bmatrix} 0 & 0 & -2 \\ 2 & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix}$$

Jordan form

$$x(t) = c_1 e^{3t} \begin{bmatrix} 0 \\ 2 \\ 1 \end{bmatrix} + c_2 e^{t} \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} + c_3 \left(e^{t} \begin{bmatrix} -2 \\ 1 \\ 0 \end{bmatrix} + t e^{t} \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \right)$$

$$\begin{bmatrix} \boxed{3} & 0 & 0 & 0 & 0 & 0 \\ & 1 & 1 & 0 & 0 & 0 \\ & 0 & 1 & 0 & 0 & 0 \\ & 0 & 0 & 2 & 1 & 0 \\ & & & & 2 & 1 \\ & 0 & 0 & & & 2 \end{bmatrix}$$