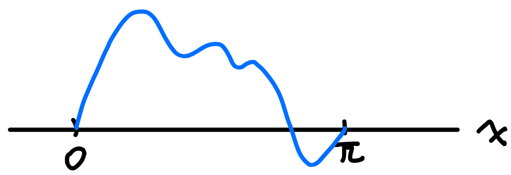
Heat eqn as an infinite linear system of ODEs

Ex:
$$\begin{cases} \dfrac{\partial u}{\partial t} = \dfrac{\partial^2 u}{\partial x^2} \\ u(0,t) = u(\pi,t) = 0 \\ u(x,0) = f(x) \end{cases}$$



Representing u(x,t) using Fourier sine series for each fixed t.

$$u(x,t) = \sum_{n=1}^{\infty} a_n(t) \sin(nx)$$

$$a_n(t) = \frac{2}{\pi}\int_0^{\pi} u(x,t)\sin(nx)\,dx \quad , \quad a_n(0) = \frac{2}{\pi}\int_0^{\pi} f(x)\sin(nx)\,dx$$

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2}$$

$$\Rightarrow \quad \frac{\partial u}{\partial t} = \sum_{n=1}^{\infty} a_n'(t) \sin(nx)$$

$$\parallel$$

$$\frac{\partial^2 u}{\partial x^2} = \sum_{n=1}^{\infty} -n^2 a_n(t) \sin(nx)$$

$$\Rightarrow \quad a_n'(t) = -n^2 a_n(t)$$

i.e. for $a(t) = \begin{bmatrix} a_1(t) \\ a_2(t) \\ a_3(t) \\ \vdots \end{bmatrix}_{\infty\text{-ly long}}$, $\quad a'(t) = D\,a(t)$

where $D = \begin{bmatrix} -1^2 & & & 0 \\ & -2^2 & & \\ & & -3^2 & \\ 0 & & & \ddots \end{bmatrix}_{\infty \times \infty}$

$$\Rightarrow \quad a(t) = e^{Dt} a(0) = \begin{bmatrix} e^{-1^2 t} a_1(0) \\ e^{-2^2 t} a_2(0) \\ e^{-3^2 t} a_3(0) \\ \vdots \end{bmatrix}$$

$$\Rightarrow \quad u(x,t) = \sum_{n=1}^{\infty} a_n(t) \sin(nx) = \sum_{n=1}^{\infty} a_n(0)\, e^{-n^2 t} \sin(nx)$$

<u>RMK</u>: If we use Fourier cosine series to represent $u(x,t)$ instead, it might appear that we get the same a_n, but not $a_n(0) = \frac{2}{\pi}\int_0^{\pi} f(x)\cos(nx)\,dx$ is different. This amounts to a phase shift for the $\cos(nx)$ which will make it equivalent to using $\sin(nx)$.

<u>Representing $u(x,t)$ using power series for each fixed t.</u>

$$u(x,t) = \sum_{n=1}^{\infty} a_n(t)\, x^n$$

(there's no need to have a $n=0$ term since at $x=0$ u needs to be 0)

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2}$$

$$\Rightarrow \quad \frac{\partial u}{\partial t} = \sum_{n=1}^{\infty} a_n'(t)\, x^n = a_1' x^1 + a_2' x^2 + a_3' x^3 + a_4' x^4 + \cdots$$

$$\parallel$$

$$\frac{\partial^2 u}{\partial x^2} = \sum_{n=2}^{\infty} a_n(t)\, n(n-1)\, x^{n-2} = 2a_2 + 6a_3 x + 12 a_3 x^2 + \cdots$$

$\Rightarrow$ for $n \geq 1$

$$a_n' = (n+2)(n+1)\, a_{n+2}$$

i.e. for $a(t) = \begin{bmatrix} a_1(t) \\ a_2(t) \\ a_3(t) \\ \vdots \end{bmatrix}_{\infty\text{-ly long}}$, $\quad a'(t) = A\, a(t)$

where $A = \begin{bmatrix} 0 & 0 & 6 & 0 & 0 & \cdots & & \\ 0 & 0 & 0 & 12 & 0 & \cdots & & \\ 0 & 0 & 0 & 0 & 20 & \cdots & & \\ & & & & & 30 & \cdots & \\ & & & & & & 42 & \cdots \\ & & & & & & \vdots & \ddots \end{bmatrix}_{\infty \times \infty}$

A is not diagonal

$$a(t) = e^{At} a(0) \quad ,$$