## Fourier Series Summary

Disclaimer: this quick summary is not a comprehensive list of everything you need to know about Fourier series. See lecture notes for more comprehensive information.

1. Consider the vector space $L^2([-L, L])$ with the inner product given by

$$\langle f, g\rangle = \int_{-L}^{L} f(x)g(x)dx.$$

Consider the set

$$\mathcal{B} = \left\{1,\ \cos\frac{\pi x}{L},\ \sin\frac{\pi x}{L},\ \cos\frac{2\pi x}{L},\ \sin\frac{2\pi x}{L},\ \cos\frac{3\pi x}{L},\ \sin\frac{3\pi x}{L}, \dots\right\}.$$

2. (Orthogonality) Functions in the above set are pairwise orthogonal with respect to the above inner product. We have

$$\int_{-L}^{L} \sin\frac{m\pi x}{L}\cos\frac{n\pi x}{L}dx = 0,$$

$$\int_{-L}^{L} \sin\frac{m\pi x}{L}\sin\frac{n\pi x}{L}dx = \begin{cases} 0, & m \neq n, \\ L, & m = n, \end{cases}$$

$$\int_{-L}^{L} \cos\frac{m\pi x}{L}\cos\frac{n\pi x}{L}dx = \begin{cases} 0, & m \neq n, \\ L, & m = n \neq 0, \\ 2L, & m = n = 0. \end{cases}$$

3. (Completeness) If $f \in L^2([-L, L])$, then its Fourier series, which is given by

$$\frac{a_0}{2} + \sum_{n=1}^{\infty}\left\{a_n \cos\frac{n\pi x}{L} + b_n \sin\frac{n\pi x}{L}\right\},$$

where

$$a_n = \frac{1}{L}\int_{-L}^{L} f(x)\cos\frac{n\pi x}{L}dx, \quad n = 0, 1, 2, \dots,$$

$$b_n = \frac{1}{L}\int_{-L}^{L} f(x)\sin\frac{n\pi x}{L}dx, \quad n = 1, 2, 3 \dots,$$

converges to $f(x)$ with respect to the $L^2$-norm. In addition,

$$\frac{a_0^2}{2} + \sum_{n=1}^{\infty}(a_n^2 + b_n^2) = \frac{1}{L}\int_{-L}^{L} f^2(x)dx = \frac{1}{L}||f||^2 \quad \text{(Parseval's identity).}$$

4. (Pointwise and uniform convergence theorems) If $f \in PS([-L, L])$, i.e. $f$, $f'$ are piecewise continuous with finitely many discontinuities at which the left and right limits exists and are finite, then the Fourier series converges pointwise to $f(x)$ at all points where $f(x)$ is continuous. At a discontinuity $x_0$, it converges to the midpoint of the jump, i.e. $\frac{1}{2}\left(\lim_{x\to x_0^-} f(x) + \lim_{x\to x_0^+} f(x)\right)$.

If $f \in PS([-L, L])$ and $f$ is continuous, then the Fourier series converges uniformly, and we can integrate its Fourier series term by term. If $f \in PS([-L, L])$ and $f''$ is also piecewise continuous, then we can differentiate its Fourier series term by term.

5. (Fourier cosine series) For $f : [0, L] \to \mathbb{R}$, its Fourier cosine series is

$$f = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\left(\frac{n\pi x}{L}\right),$$

where

$$a_n = \frac{2}{L} \int_0^L f(x) \cos\left(\frac{n\pi x}{L}\right) dx, \quad n = 0, 1, 2, 3, \ldots$$

6. (Fourier sine series) For $f : [0, L] \to \mathbb{R}$, its Fourier sine series is

$$f = \sum_{n=1}^{\infty} b_n \sin\left(\frac{n\pi x}{L}\right),$$

where

$$b_n = \frac{2}{L} \int_0^L f(x) \sin\left(\frac{n\pi x}{L}\right) dx. \quad n = 1, 2, 3, \ldots$$