Intro to heat equation

Heat eqn $\frac{\partial u}{\partial t} - \alpha^2 \frac{\partial^2 u}{\partial x^2} = 0$ (parabolic)

Wave eqn $\frac{\partial^2 u}{\partial t^2} - a^2 \frac{\partial^2 u}{\partial x^2} = 0$ (hyperbolic)

Laplace eqn $\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$ (elliptic)

Reading: "Preliminary discourse" in Fourier's book "Analytical Theory of Heat" 1878.

Remark: heat flow is irreversible

Reason: If $u(x,t)$ is a soln to $\frac{\partial u}{\partial t} = \alpha^2 \frac{\partial^2 u}{\partial x^2}$,

then $\tilde{u}(x,t) = u(x,-t)$ is not a soln to $\frac{\partial \tilde{u}}{\partial t} = \alpha^2 \frac{\partial^2 \tilde{u}}{\partial x^2}$.

(b/c $\frac{\partial \tilde{u}}{\partial t} = -\frac{\partial u}{\partial t} = -\alpha^2 \frac{\partial^2 u}{\partial x^2} = -\alpha^2 \frac{\partial \tilde{u}}{\partial x^2}$)

Some related equations to heat eqn's

Schrödinger eqn: $i\hbar \frac{\partial \psi(x,t)}{\partial t} = -\frac{\hbar^2}{2m} \frac{\partial^2 \psi(x,t)}{\partial x^2}$

Black-Scholes eqn: $\frac{\partial V}{\partial t} + \frac{1}{2}\sigma^2 s^2 \frac{\partial^2 V}{\partial s^2} = rV - rs\frac{\partial V}{\partial s}$

"The pricing of options and Corporate liability" Journal of Political Economy 1973.

Random walk of a particle along the real line.

At each time step of size h, the particle jumps left or right by a distance of r, with probability $\frac{1}{2}$.

At $t=0$, $x=x_0$,

$u(x,t)$ = probalility density of the location of the particle in the limit $h, r \to 0$.

i.e. the probality of finding the particle in $[a,b]$ at t is $\int_a^b u(x,t)\, dt$

$u(x,t+h)$ = probality of reaching location x at time $t+h$.

$$u(x,t+h) = \frac{1}{2} u(x-r,t) + \frac{1}{2} u(x+r,t)$$

Taylor expansion
$$\begin{cases} u(x,t+h) \approx u(x,t) + \frac{\partial u}{\partial t}(x,t)\, h + \cdots \\ u(x \pm r,t) \approx u(x,t) \pm \frac{\partial u}{\partial x}(x,t)\, r + \frac{1}{2}\frac{\partial^2 u}{\partial x^2}(x,t)\, r^2 + \cdots \end{cases}$$

$$\Rightarrow \quad \frac{\partial u}{\partial t}(x,t)\, h + \cdots = \frac{r^2}{2}\frac{\partial^2 u}{\partial x^2} + \cdots$$

in the limit $r, h \to 0$, suppose $\frac{1}{2}\frac{r^2}{h} \to \alpha^2$

$$\Rightarrow \quad \frac{\partial u(x,t)}{\partial t} = \alpha^2 \frac{\partial^2 u}{\partial x^2}$$