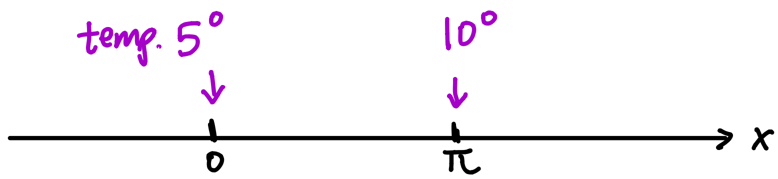
Heat eqn with nonhomogeneous boundary conditions

Ex: $\dfrac{\partial u}{\partial t} = 2\dfrac{\partial^2 u}{\partial x^2}$, $0<x<\pi$, $t>0$

boundary cond: $u(0,t)=5$, $u(\pi,t)=10$

Initial cond: $u(x,0)=f(x)$



Solution

Equilibrium soln, i.e. soln $u_0(x)$ that doesn't change with time (no diffusion).

$$0 = \frac{\partial u_0}{\partial t} = \frac{\partial^2 u_0(x)}{\partial x^2} = 0 \quad \Rightarrow \quad u_0(x) = ax+b$$

$$u_0(0)=5, \quad u_0(\pi)=10 \quad \Rightarrow \quad u_0(x) = 5 + \frac{5}{\pi}x$$

Any other solution:

$$u(x,t) = u_0(x) + \tilde{u}(x,t)$$

$$\tilde{u}(x,t) = u(x,t) - u_0(x)$$

$$\left.\begin{aligned} \frac{\partial \tilde{u}}{\partial t} &= \frac{\partial u}{\partial t} - \frac{\partial u_0}{\partial t} = \frac{\partial u}{\partial t} \\ \frac{\partial^2 \tilde{u}}{\partial x^2} &= \frac{\partial^2 u}{\partial x^2} - \frac{\partial^2 u_0}{\partial x^2} = \frac{\partial^2 u}{\partial x^2} \end{aligned}\right\} \Rightarrow \boxed{\frac{\partial \tilde{u}}{\partial t} = 2\frac{\partial^2 \tilde{u}}{\partial x^2}}$$

bdry cond. for $\tilde{u}$: $\tilde{u}(0,t) = \tilde{u}(\pi,t) = 0$

initial cond. for $\tilde{u}$: $\tilde{u}(x,0) = u(x,0) - u_0(x)$

$$= f(x) - \left(5 + \frac{5}{\pi}x\right)$$

Find $\tilde{u}(x,t)$

Then $u(x,t) = \tilde{u}(x,t) + u_0(x)$