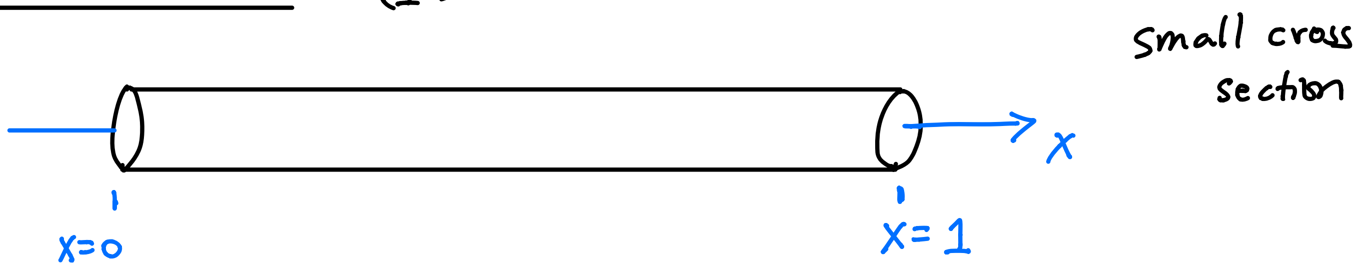
The heat equation

<u>Simpliest example</u> (1 D metal rod)



$u(x,t)$ = temperature

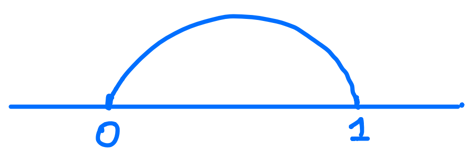
$$\frac{\partial u}{\partial t} = \alpha^2 \underbrace{\frac{\partial^2 u}{\partial x^2}}, \quad 0 \le x \le 1, \quad t > 0$$

difference between $u(x,y)$ and the average of the nearby pts.

for this ex: $\frac{\partial u}{\partial t} = 2\frac{\partial^2 u}{\partial x^2}$

Boundary value: $u(0,t) = 0$, $u(1,t) = 0$

Initial value: $u(x,0) = f(x) = \sin(\pi x)$



<u>Seperation variables</u>: $u(x,t) = V(x)\,w(t)$

$$\frac{\partial u}{\partial t} = 2\frac{\partial^2 u}{\partial x^2} \iff V(x)\frac{dw(t)}{dt} = 2\,w(t)\frac{d^2 V(x)}{dx}$$

dividing $V(x)\,w(t)$ on both sides, we get

$$\underbrace{\frac{1}{2w(t)}\frac{dw(t)}{dt}}_{\text{function of } t} = \underbrace{\frac{1}{v(x)}\frac{d^2v(x)}{dx^2}}_{\text{function of } x}$$

So both sides equal to a constant

$$\frac{1}{2w(t)}\frac{dw(t)}{dt} = \lambda = \frac{1}{v(x)}\frac{d^2v(x)}{dx^2}$$

i.e. we have replaced the above PDE with two ODEs

$$\begin{cases} \dfrac{dw(t)}{dt} = 2\lambda w(t) \\[2ex] \dfrac{d^2v(x)}{dx^2} = \lambda v(x) \end{cases}$$

<u>Solve for w(t)</u>

$$\frac{dw(t)}{dt} = 2\lambda\, w(t) \quad \Rightarrow \quad w(t) = C e^{2\lambda t}$$

As $t \to \infty$, we are cooling down so $u \to 0$, so $w \to 0$

so physical intuition $\Rightarrow$ $\lambda < 0$

We'll see math also tells us that.

<u>Solve v(x)</u>

$$v''(x) = \lambda v(x)$$

$$0 = u(0,t) = v(0)\, w(t) \quad \Rightarrow \quad \boxed{v(0) = 0}$$

$$0 = u(1,t) = v(1)\, w(t) \quad \Rightarrow \quad \boxed{v(1) = 0}$$

char: $r^2 = \lambda$, $\quad r = \pm\sqrt{\lambda}$

$\lambda > 0$: $v = c_1 e^{\sqrt{\lambda}x} + c_2 e^{-\sqrt{\lambda}x}$

$$\left.\begin{aligned} 0 = v(0) &= c_1 + c_2 \\ 0 = v(1) &= c_1 e^{\sqrt{\lambda}} + c_2 e^{-\sqrt{\lambda}} \end{aligned}\right\} \underbrace{\begin{bmatrix} 1 & 1 \\ e^{\sqrt{\lambda}} & e^{-\sqrt{\lambda}} \end{bmatrix}}_{\det \neq 0} \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\Rightarrow c_1 = 0 = c_2 \Rightarrow v = 0$$

$\lambda = 0$: $v''(x) = 0 \quad \Rightarrow \quad v = c_1 x + c_2$

$$\left.\begin{aligned} 0 = v(0) &= c_2 \\ 0 = v(1) &= c_1 \end{aligned}\right\} \Rightarrow v = 0$$

$\lambda < 0$ $\quad v = c_1 \cos(\sqrt{-\lambda}\,x) + c_2 \sin(\sqrt{-\lambda}\,x)$

$0 = v(0) = c_1$

$\Rightarrow v(x) = c_2 \sin(\sqrt{-\lambda}\,x)$

$0 = v(1) = c_2 \sin\sqrt{-\lambda}$

If $\sin\sqrt{-\lambda} \neq 0$, then $c_2 = 0 \Rightarrow v = 0$

If $\underbrace{\sin\sqrt{-\lambda} = 0}$, then c_2 arbitrary, $v(x) = c_2 \sin\sqrt{-\lambda}\,x$

$\Rightarrow \sqrt{-\lambda} = n\pi$, $\quad n = 1, 2, 3, 4, \ldots$

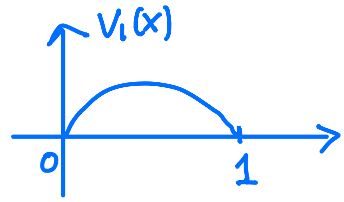
$\lambda = -n^2\pi^2$

eigenvalues $\longleftrightarrow$ eigenfunction

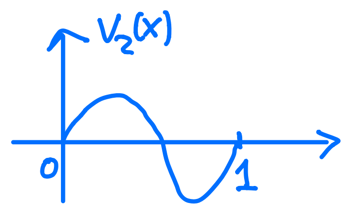
$\lambda_n = -n^2\pi^2$ $\qquad$ $V_n = \sin(n\pi x)$

$n = 1, 2, 3, \ldots$ $\qquad$ (or any constant multiples of this)

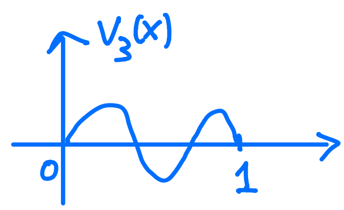
Note: $V_1(x) = \sin(\pi x)$



$V_2(x) = \sin(2\pi x)$



$V_3(x) = \sin(3\pi x)$



Collection of all separate solution

$$\left\{ V_n(x) w_n(t) = b_n e^{-2n^2\pi^2 t} \sin(n\pi x) \quad , \quad n = 1, 2, 3 \ldots \right\}$$

$$V_n(x) W_n(0) = b_n \sin(n\pi x)$$

Initial condition : $u(x,0) = \sin(\pi x)$

One of them fits the initial condition

$n=1$ $\quad$ $b_n = 1$

So $\boxed{u(x,t) = e^{-2\pi^2 t} \sin(\pi x)}$ is a soln to heat eqn + bdry condition

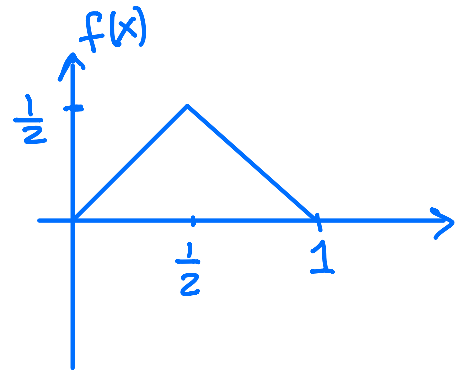
s.t. $u(x,0) = \sin(\pi x)$

Thm: heat eqn on an interval with suitable initial and boundary conditions has a unique solution.

Ex 2: $\dfrac{\partial u}{\partial t} = 2\dfrac{\partial^2 u}{\partial x^2}$, $\quad 0 \le x \le 1$

$u(0,t) = u(1,t) = 0$

$$u(x,0) = f(x) = \begin{cases} x, & 0 \le x \le \frac{1}{2} \\ 1-x, & \frac{1}{2} \le x \le 1 \end{cases}$$



Hope: there is a

$$u(x,t) = \sum_{n=1}^{\infty} b_n e^{-2n^2\pi t} \sin(n\pi x)$$

that satisfies $u(x,0) = f(x)$

$$f(x) = u(x,0) = \sum_{n=1}^{\infty} b_n \sin(n\pi x)$$

Fourier sine series for $f(x):[0,1]\to\mathbb{R}$

$$b_n = 2\int_0^1 f(x)\sin(n\pi x)\,dx = 2\left(\int_0^{1/2} x\sin(n\pi x)\,dx + \int_{\frac{1}{2}}^1 (1-x)\sin(n\pi x)\,dx\right)$$

$$=2\left(\frac{-x}{n\pi}\cos(n\pi x)\Big|_0^{\frac{1}{2}} + \int_0^{\frac{1}{2}}\frac{\cos n\pi x}{n\pi}\,dx + \frac{-(1-x)}{n\pi}\cos(n\pi x)\Big|_{\frac{1}{2}}^1 - \int_{\frac{1}{2}}^1 \frac{\cos n\pi x}{n\pi}\,dx\right)$$

$$= 2\left(-\frac{\cos\left(\frac{n\pi}{2}\right)}{2n\pi} + \frac{1}{(n\pi)^2}\sin(n\pi x)\Big|_0^{\frac{1}{2}} + \frac{1}{2n\pi}\cos\left(\frac{n\pi}{2}\right) - \frac{1}{(n\pi)^2}\sin(n\pi x)\Big|_{\frac{1}{2}}^1\right)$$

$$= 2\left(\frac{2}{n^2\pi^2}\sin\left(\frac{n\pi}{2}\right)\right)$$

$$\sin\left(\frac{n\pi}{2}\right) = \begin{cases} 0 & \text{if } n \text{ even} \\ 1 & \text{if } n = 1, 5, 9, \dots \\ -1 & \text{if } n = 3, 7, 11, \dots \end{cases}$$

$$u(x,t) = \sum_{n=1}^{\infty} \frac{4}{n^2\pi^2}\sin\left(\frac{n\pi}{2}\right) e^{-2n^2\pi^2 t}\sin(n\pi x)$$