Given $f : [-\pi, \pi] \longrightarrow \mathbb{R}$, (For simplicity, we took $L = \pi$)

its Fourier series can be written as

$$f(x) = \sum_{n=-\infty}^{\infty} c_n e^{inx}$$

where $C_n = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(x) e^{inx}$.

$$e^{inx} = \cos nx + i \sin nx$$

$$\Rightarrow C_n e^{inx} + C_{-n} e^{-inx} = C_n \cos nx + i C_n \sin nx + C_{-n} \cos nx - i C_{-n} \sin nx$$

$$= a_n \cos nx + b_n \sin nx$$

$a_n = C_n + C_{-n}$ $\qquad$ $b_n = i(C_n - C_{-n})$

$$\Rightarrow f(x) = \sum_{n=-\infty}^{\infty} C_n e^{inx} = \frac{a_0}{2} + \sum_{n=1}^{\infty} (a_n \cos nx + b_n \sin nx).$$

$$f(x),\ x \in [-\pi, \pi] \xrightarrow{\text{encode it as}} \hat{f}(n) = C_n,\ n \in \mathbb{Z}.$$

Fourier transform

What about $f : \mathbb{R} \longrightarrow \mathbb{R}$?

Not enough to represent f only using e^{inx}, $n \in \mathbb{Z}$

need all of e^{ikx}, $k \in \mathbb{R}$.

$$f(x),\ x \in \mathbb{R} \xrightarrow[\text{Fourier Transform}]{\text{encode it as}} \hat{f}(k),\ k \in \mathbb{R}$$

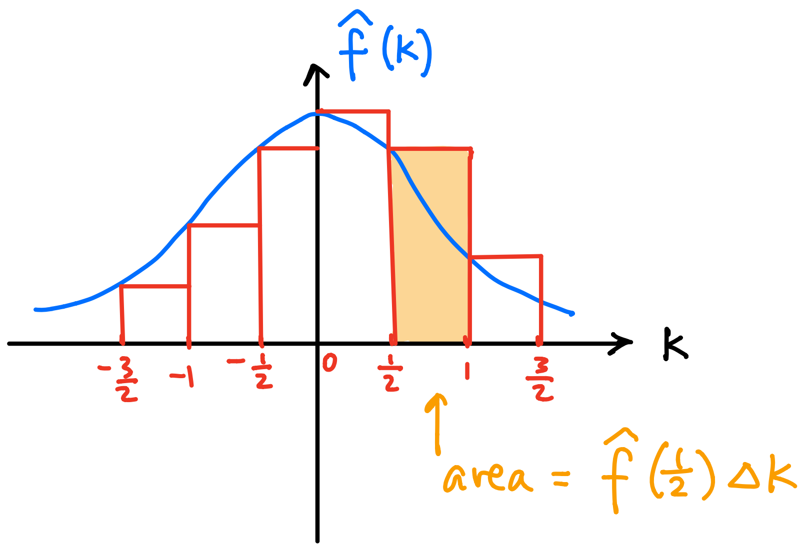
$$f(x) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \hat{f}(k)\, e^{ikx}\, dk \qquad \hat{f}(k) = \int_{-\infty}^{\infty} f(x) e^{-ikx}\, dx$$

Using this formula to appoximate $f(x)$ using $e^{i(\text{half integers})x}$

Let $\Delta k = \frac{1}{2}$

$$f(x) \approx \left(\tfrac{1}{2\pi}\hat{f}(0)\Delta k\right) e^{i0x} + \left(\tfrac{1}{2\pi}\hat{f}(\tfrac{1}{2})\Delta k\right) e^{i\frac{1}{2}x} + \left(\tfrac{1}{2\pi}\hat{f}(-\tfrac{1}{2})\Delta k\right) e^{-i\frac{1}{2}x}$$
$$+ \left(\tfrac{1}{2\pi}\hat{f}(1)\Delta k\right) e^{ix} + \left(\tfrac{1}{2\pi}\hat{f}(-1)\Delta k\right) e^{-ix} + \cdots$$

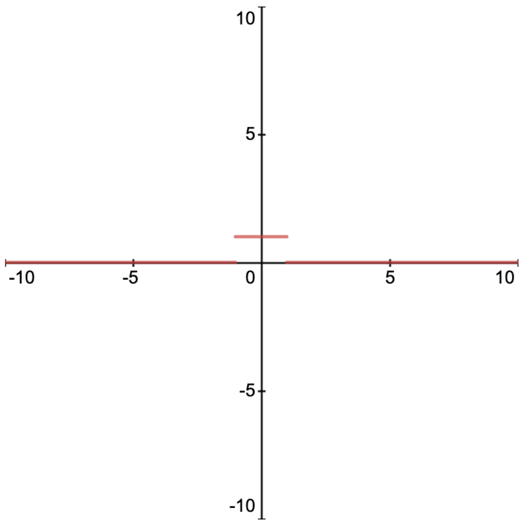
Suppose we know what $\hat{f}(k)$ is

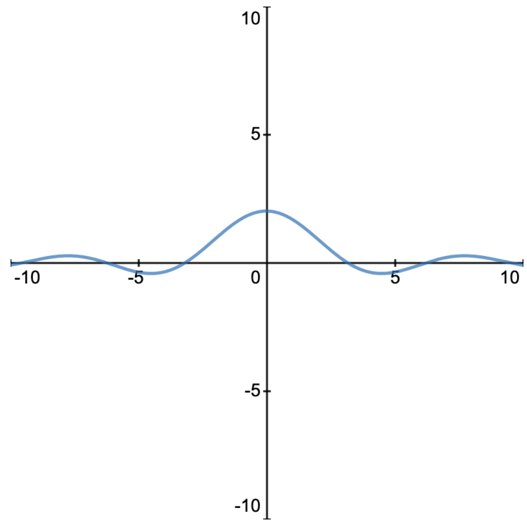


Ex 1

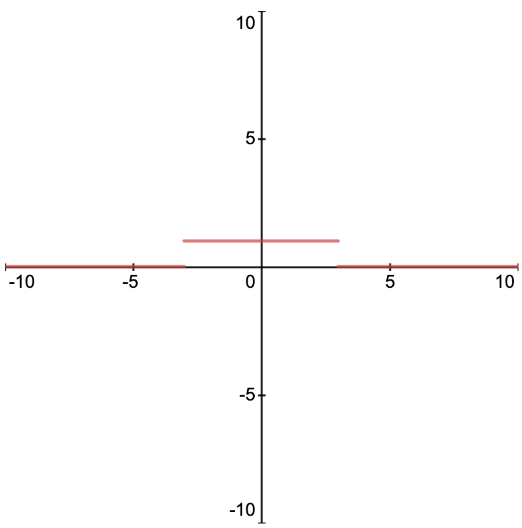
$$f_a(x) = \begin{cases} 1 & \text{if } x \in (-a, a) \\ 0 & \text{otherwise} \end{cases} \quad , \quad \hat{f}_a(k) = \frac{2a \sin(ak)}{ax}$$

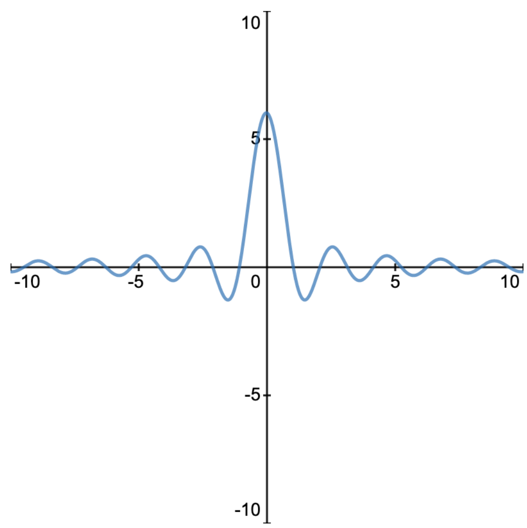
$a = 1$





$a = 3$





$\Delta x \, \Delta k$ stays the same (uncertainty principle)

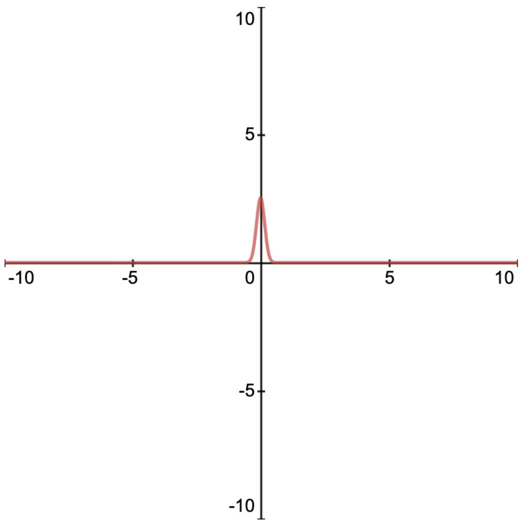
② Gaussian

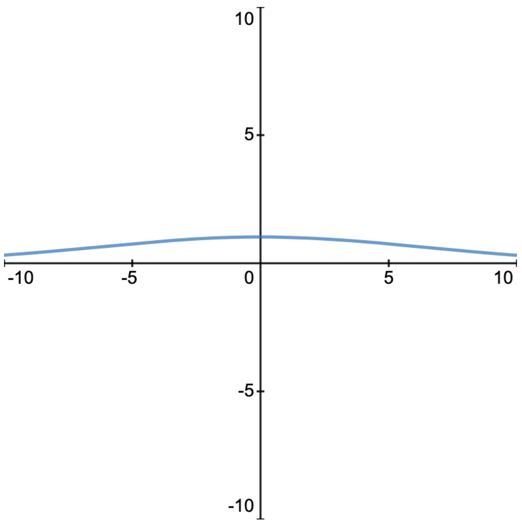
$$\hat{f}_a(x) = \sqrt{\frac{a}{\pi}}\, e^{-ax^2} \quad , \quad \hat{f}_a(k) = e^{-\frac{k^2}{4a}}$$

chosen so

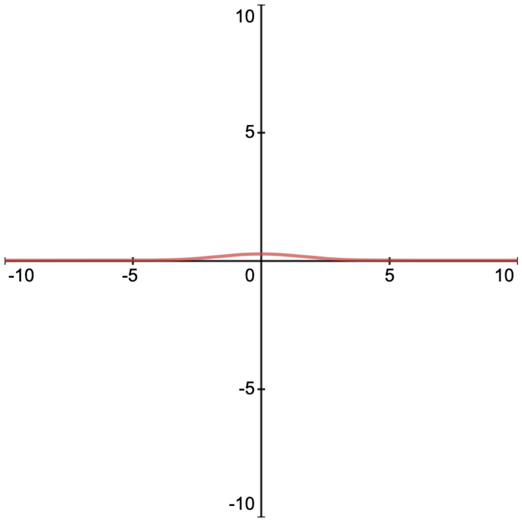
$$\int_{-\infty}^{\infty} f_a(x)\, dx = 1$$

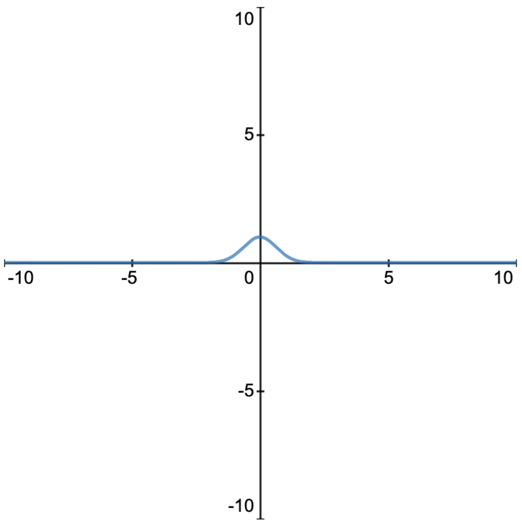
$a = 20$





$a = 0.2$





As $a \to \infty$, $f_a(x) \to \delta(x)$

Let $a = \frac{1}{4\alpha^2 t}$

$$G(x,t) = \frac{1}{\sqrt{4\pi\alpha^2 t}} e^{-\frac{x^2}{4\alpha^2 t}} \quad , \qquad \hat{G}(k,t) = e^{-k^2\alpha^2 t}$$

Solves the heat equation

$$\begin{cases} \frac{\partial G}{\partial t} = \alpha^2 \frac{\partial^2 G}{\partial x^2}, & x \in \mathbb{R} \\ G(x,0) = \delta(x) \end{cases}$$

Solution to the heat equation

$$\begin{cases} \frac{\partial u}{\partial t} = \alpha^2 \frac{\partial^2 u}{\partial x^2}, & x \in \mathbb{R} \\ u(x,0) = f(x) \end{cases}$$

is the convolution

$$u(x,t) = G(x,t) * f(x) = \int_{-\infty}^{\infty} G(x-\xi,t) f(\xi)\, d\xi$$

$$u(x,0) = \int_{-\infty}^{\infty} \delta(x-\xi) f(\xi)\, d\xi = f(x)$$

Perspective using Fourier transform

$$\begin{cases} \text{Heat equation} \\ \dfrac{\partial u}{\partial t} = \alpha^2 \dfrac{\partial^2 u}{\partial x^2}, \quad x \in \mathbb{R} \\ u(x,0) = f(x) \end{cases}$$

$$F\{u(x,t)\} = \hat{u}(k,t)$$

$$F\{u_x(t)\} = ik\,\hat{u}(k,t)$$

$$F\{u_{xx}(t)\} = -k^2\,\hat{u}(k,t)$$

$$\frac{\partial u}{\partial t} = \alpha^2 \frac{\partial^2 u}{\partial x^2} \xrightarrow{\text{Fourier transform}} \frac{d}{dt}\hat{u}(k,t) = -k^2\alpha^2\,\hat{u}(k,t)$$

$$u(x,t) = G(x,t) * u(x,0) \xleftarrow{(\text{Fourier transform})^{-1}} \hat{u}(k,t) = \underbrace{e^{-k^2\alpha^2 t}}_{=\hat{G}(k,t)}\,\hat{u}(k,0)$$