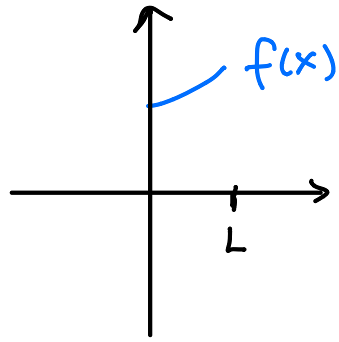
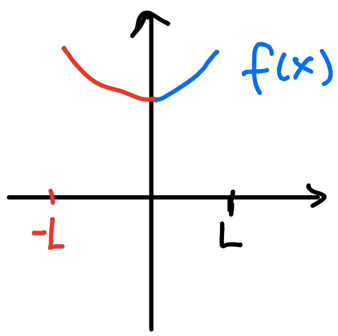
Half interval Fourier series

Given $f : [0, L] \to \mathbb{R}$



Even completion



$f(x) : [0, L] \to \mathbb{R}$

$\Downarrow$

$f^{even} : [-L, L] \to \mathbb{R}$

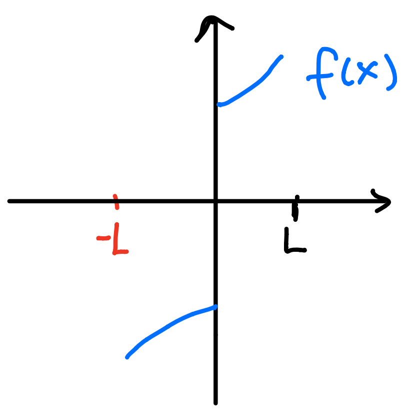
$f^{even}(x) = f^{even}(-x)$

Fourier cosine series for $f(x)$

= Fourier series for the even completion $f^{even}(x)$

$$\text{On } [0,L], \quad f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos \frac{n\pi x}{L}$$

odd completion



$f(x) : [0, L] \to \mathbb{R}$

$\Downarrow$

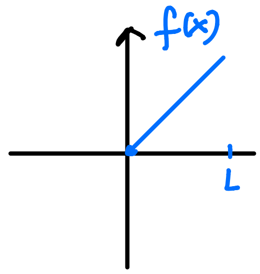
$f^{odd}(x) : [-L, L] \to \mathbb{R}$

$f^{odd}(x) = -f^{odd}(-x)$

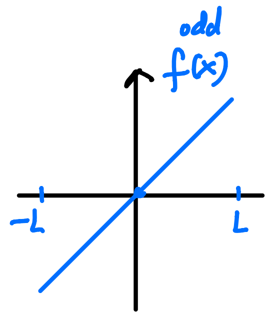
Fourier sine series of $f(x)$

$= $ Fourier series of the odd completion $f^{odd}(x)$

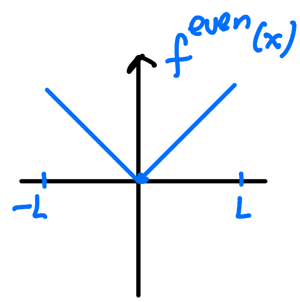
on $[0,L]$, $f(x) = \sum_{n=1}^{\infty} b_n \sin \frac{n\pi x}{L}$



Ex $f(x) = x$ for $x \in [0,L]$



$f^{odd}(x) = x$ for $x \in [-L, L]$



$f^{even}(x) = |x|$

$$= \begin{cases} x & \text{on } [0,L] \\ -x & \text{on } [-L,0] \end{cases}$$

Fourier sine series for $f(x) = \sum_{n=1}^{\infty} b_n \sin \frac{n\pi x}{L}$

$$b_n = \frac{1}{L} \int_{-L}^{L} f^{odd}(x) \sin \frac{n\pi x}{L} \, dx$$

$$\boxed{b_n = \frac{2}{L} \int_0^L f(x) \sin \frac{n\pi x}{L} \, dx}$$

$$f(x) = \frac{2L}{\pi} \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n} \sin \frac{n\pi x}{L}$$ (computed before)

Fourier cosine series for $f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\frac{n\pi x}{L}$

$$a_n = \frac{1}{L}\int_{-L}^{L} f^{even}(x) \cos\frac{n\pi x}{L}\, dx$$

$$\boxed{a_n = \frac{2}{L}\int_0^L f(x) \cos\frac{n\pi x}{L}\, dx}$$

$$a_n = \frac{2}{L}\int_0^L x \cos\frac{n\pi x}{L}\, dx$$

$$\begin{cases} \text{if } n=0, \text{ then } a_0 = \frac{2}{L}\int_0^L x\, dx = \frac{2}{L}\frac{x^2}{2}\Big|_0^L = L \\ \text{if } n\neq 0, \text{ then see below} \end{cases}$$

$$\underset{n\neq 0}{=} \frac{2}{L}\left(\underbrace{\frac{L}{n\pi} x \sin\frac{n\pi x}{L}\Big|_0^L}_{=0} - \frac{L}{n\pi}\int_0^L \sin\frac{n\pi x}{L}\, dx\right)$$

$$= \frac{2}{L}\left(\left(\frac{L}{n\pi}\right)^2 \cos\frac{n\pi x}{L}\Big|_0^L\right)$$

$$= \frac{2L}{(n\pi)^2}\left((-1)^n - 1\right)$$

$$= \begin{cases} 0 & \text{if } n \text{ even} \\ -\frac{4L}{(n\pi)^2} & \text{if } n \text{ odd} \end{cases}$$

$$f(x) = \frac{L}{2} - \frac{4L}{\pi^2}\sum_{m=1}^{\infty} \frac{1}{(2m-1)^2} \cos\left(\frac{(2m-1)\pi x}{L}\right)$$