Fourier series

$f(t)$ "=" $\sum$ sin / cos of different frequencies

Example of a Hilbert space

$$L^2([-L,L]) = \left\{ f : [-L,L] \to \mathbb{R} \ \text{ s.t. } \int_{-L}^{L} |f(x)|^2 \, dx < \infty \right\}$$

or can think of $f: \mathbb{R} \to \mathbb{R}$ and $2L$ periodic

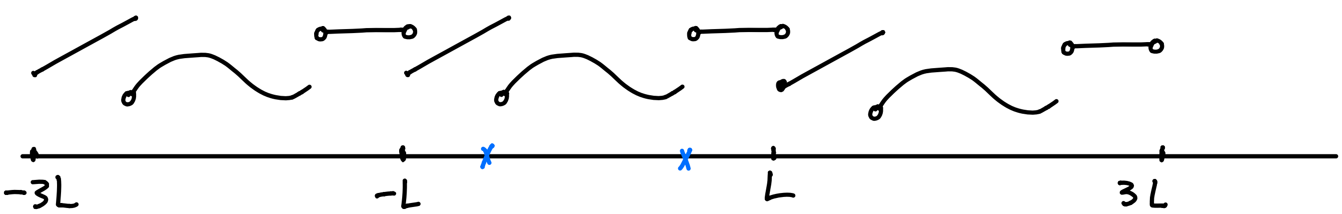
Lebesgue integration, but for most functions, same as Riemann integration

A useful subset $PS([-L,L]) \subseteq L^2([-L,L])$

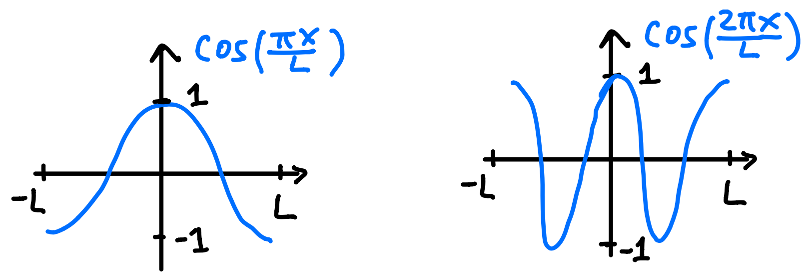
"piecewise smooth", not exactly a standard terminology

Precisely

$$PS([-L,L]) = \left\{ f: [-L,L] \to \mathbb{R} \ \text{ s.t. } f, f' \text{ are piecewise continuous with finitely many discontinuities at which the left and right limits exist and are finite.} \right\}$$



<u>Ex 1</u>: $f(x) = \cos\left(\overbrace{\frac{2\pi n}{2L}}^{K} x\right) = \cos\left(\frac{n\pi x}{L}\right), \ n \in \mathbb{Z}$

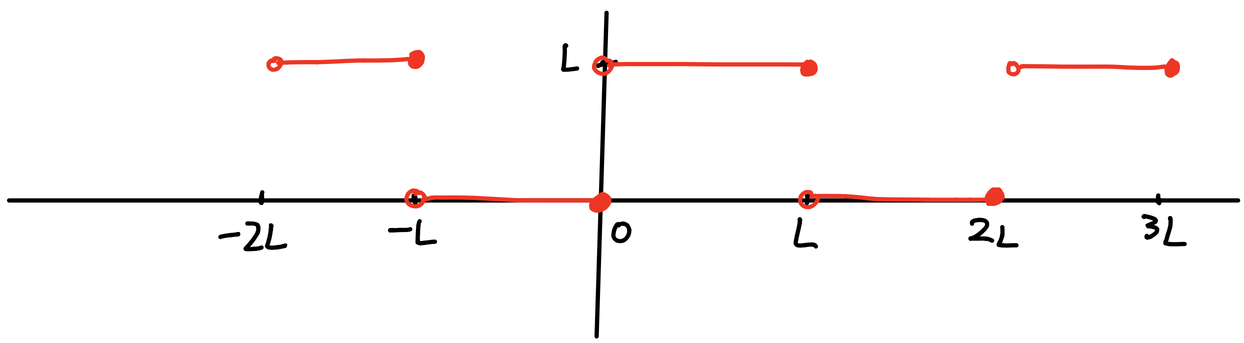


$k = \frac{n\pi}{L}$ angular frequency

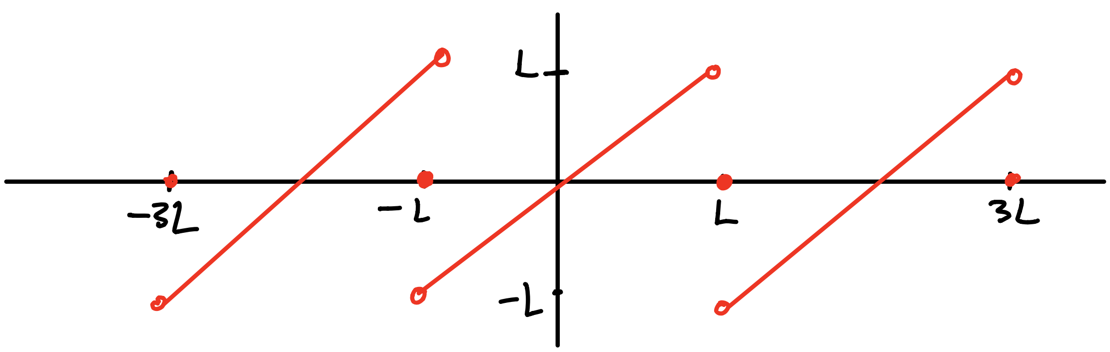
$\frac{2\pi}{k}$ = period

Ex 2 : $f(x) = \sin\left(\frac{n\pi x}{L}\right)$, $n \in \mathbb{Z}$

Ex 3 : $f(x) = \begin{cases} 0, & -L < x \leq 0 \\ L, & 0 < x \leq L \end{cases}$



Ex 4 : $f(x) = \begin{cases} x, & -L < x < L \\ 0, & x = -L \end{cases}$



Can check : $L^2([-L,L])$ is a vector space.

Inner product on $L^2([-L,L]) \ni g, h$

$$\langle g, h \rangle := \int_{-L}^{L} g(x) h(x)\, dx$$

L^2 norm : for $f \in L^2([-L,L])$, $\|f\| = \left(\int_{-L}^{L} |f(x)|^2 d \right)^{1/2}$

Consider the set

$$B = \left\{ 1, \cos\frac{\pi x}{L}, \sin\frac{\pi x}{L}, \cos\frac{2\pi x}{L}, \sin\frac{2\pi x}{L}, \dots \right\}$$

will see that B is an orthogonal basis for $L^2([-L,L])$

Claim (orthogonality) any pair of functions in this set are orthogonal w.r.t. the above inner product.

Indeed

$$\left\langle \sin\frac{m\pi x}{L}, \cos\frac{n\pi x}{L} \right\rangle = \int_{-L}^{L} \sin\frac{m\pi x}{L} \cos\frac{n\pi x}{L}\, dx = 0$$

$$\left\langle \sin\frac{m\pi x}{L}, \sin\frac{n\pi x}{L} \right\rangle = \int_{-L}^{L} \sin\frac{m\pi x}{L} \sin\frac{n\pi x}{L}\, dx = \begin{cases} 0, & m \neq n \\ L, & m = n \end{cases}$$

$$\left\langle \cos\frac{m\pi x}{L}, \cos\frac{n\pi x}{L} \right\rangle = \int_{-L}^{L} \cos\frac{m\pi x}{L} \cos\frac{n\pi x}{L}\, dx = \begin{cases} 0, & m \neq n \\ L, & m = n \neq 0 \\ 2L, & m = n = 0 \end{cases}$$

Suppose

$$f(x) = \underbrace{\frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\frac{n\pi x}{L}}_{\text{even}} + \underbrace{\sum_{n=1}^{\infty} b_n \sin\frac{n\pi x}{L}}_{\text{odd}},$$

(even g: $g(x) = g(-x)$) (odd g: $g(-x) = -g(x)$)

let's see what a_n, b_n needs to be.

Note: In ex 4, $f(x)$ is odd, so $a_n = 0$ for $n = 0, 1, 2, 3 \ldots$

$$\underset{n=1,2,3\ldots}{a_n} = \frac{\langle f, \cos\frac{n\pi x}{L}\rangle}{\langle \cos\frac{n\pi x}{L}, \cos\frac{n\pi x}{L}\rangle} = \frac{1}{L}\int_{-L}^{L} f(x)\cos\frac{n\pi x}{L}\,dx$$

same with vectors in $\mathbb{R}^2$, $v = v_1 e_1 + v_2 e_2$, $e_1 \cdot e_2 = 0$

$v \cdot e_1 = v_1 e_1 \cdot e_1$, $v_1 = \frac{v_1 \cdot e_1}{e_1 \cdot e_1}$

also $$\frac{a_0}{2} = \frac{\langle f, \cos\frac{0\pi x}{L}\rangle}{\langle \cos\frac{0\pi x}{L}, \cos\frac{0\pi x}{L}\rangle} = \frac{1}{2L}\int_{-L}^{L} f(x)\underbrace{\cos\frac{0\pi x}{L}}_{=1}\,dx$$

So $$\underset{n=0,1,2,3,\ldots}{a_n} = \frac{1}{L}\int_{-L}^{L} f(x)\cos\frac{n\pi x}{L}\,dx$$

Similarly

$$\underset{n=1,2,3\ldots}{b_n} = \frac{\langle f, \sin\frac{n\pi x}{L}\rangle}{\langle \sin\frac{n\pi x}{L}, \sin\frac{n\pi x}{L}\rangle} = \frac{1}{L}\int_{-L}^{L} f(x)\sin\frac{n\pi x}{L}\,dx.$$

Also can check:

$$\frac{1}{L}\int_{-L}^{L} f(x)\cos\frac{n\pi x}{L}\,dx$$

$$= \frac{1}{L}\int_{-L}^{L}\left(\frac{a_0}{2} + \sum_{k=1}^{\infty} a_k\cos\frac{k\pi x}{L} + \sum_{k=1}^{\infty} b_k\sin\frac{k\pi x}{L}\right)\cos\frac{n\pi x}{L}\,dx$$

$$= \frac{1}{L}\int_{-L}^{L}\left(\frac{a_0}{2}\cos\frac{n\pi x}{L} + \sum_{k=1}^{\infty} a_k\cos\frac{k\pi x}{L}\cos\frac{n\pi x}{L} + \sum_{k=1}^{\infty} b_k\sin\frac{k\pi x}{L}\cos\frac{n\pi x}{L}\right)dx$$

$$= \frac{1}{L}\left[\int_{-L}^{L}\frac{a_0}{2}\cos\frac{n\pi x}{L}\,dx + \sum_{k=1}^{\infty}\int_{-L}^{L} a_k\cos\frac{k\pi x}{L}\cos\frac{n\pi x}{L}\,dx + \sum_{k=1}^{\infty}\int_{-L}^{L} b_k\sin\frac{k\pi x}{L}\cos\frac{n\pi x}{L}\,dx\right]$$

$$= \frac{1}{L}\int_{-L}^{L} a_n\cos\frac{n\pi x}{L}\cos\frac{n\pi x}{L}\,dx \quad = \frac{1}{L}a_n L \qquad (n=1,2,3\ldots)$$

$$= a_n$$

Parseval's identity ("Pythagorean thm")

$$\frac{1}{L}\langle f,f\rangle = \frac{\|f\|^2}{L} = \frac{1}{L}\int_{-L}^{L} f(x)^2\,dx = \frac{a_0^2}{2} + \sum_{n=1}^{\infty}(a_n^2+b_n^2)$$

Note: $a_n^2+b_n^2 = \text{amplitude}^2$ of the wave

$$a_n\cos\frac{n\pi x}{L} + b_n\sin\frac{n\pi x}{L} = \sqrt{a_n^2+b_n^2}\,\cos\left(\frac{n\pi x}{L} - \phi\right)$$

of angular frequency $\frac{n\pi}{L}$.

In the above, by writing $f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty}\left(a_n\cos\frac{n\pi x}{L} + b_n\sin\frac{n\pi x}{L}\right)$, we have encoded $f(x)$ using the sequence of numbers $\{a_n\}_{n=0}^{\infty}, \{b_n\}_{n=1}^{\infty}$.

In other words, we have written $f(x)$ as a function

$$\hat{f}(n) = (a_n, b_n), \qquad n = 0, 1, 2, 3\ldots$$

For $f \in L^2([-L,L])$, its Fourier series is defined to be

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty}\left(a_n \cos\frac{n\pi x}{L} + b_n \sin\frac{n\pi x}{L}\right)$$

will discuss soon that this might not be exactly equal for all $f \in L^2([-L,L])$, but "close", so we'll use the "=" sign.

when $a_n = \frac{1}{L}\int_{-L}^{L} f(x)\cos\frac{n\pi x}{L}\,dx$, $n = 0, 1, 2, 3, \ldots$

$b_n = \frac{1}{L}\int_{-L}^{L} f(x)\sin\frac{n\pi x}{L}\,dx$, $n = 1, 2, 3, \ldots$

Ex 4: For the $f(x)$ in Ex 4, find its Fourier series.

$a_n = 0$, $n = 0, 1, 2, \ldots$

$$b_n = \frac{1}{L}\int_{-L}^{L} x \sin\frac{n\pi x}{L}\,dx.$$

← product of odd functions is even

$\int_{-L}^{L}$ even fn $= 2\int_0^L$

$$= \frac{2}{L}\int_0^L x \sin\frac{n\pi x}{L}\,dx$$

$$= \frac{2}{L}\left(-\left(\frac{L}{n\pi}\right) x \cos\frac{n\pi x}{L}\Big|_0^L + \int_0^L \left(\frac{L}{n\pi}\right)\cos\frac{n\pi x}{L}\,dx\right)$$

$$= \frac{2}{L}\left(-\left(\frac{L}{n\pi}\right) L\cos(n\pi) + \left(\frac{L}{n\pi}\right)^2 \sin\frac{n\pi x}{L}\Big|_0^L\right)$$

$$= \frac{2L}{n\pi}(-1)^{n+1}, \quad n = 1, 2, 3, \ldots$$

$$f(x) = \frac{2L}{\pi}\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{n}\sin\frac{n\pi x}{L}$$

Ex3 For the f(x) in Ex3, find its Fourier series.

$$f(x) = \begin{cases} 0, & -L < x \leq 0 \\ L, & 0 < x \leq L \end{cases}$$

$$\underset{n=0,1,2,3\ldots}{a_n} = \frac{1}{L}\int_{-L}^{L} f(x)\cos\frac{n\pi x}{L}\,dx$$

$$= \frac{1}{L}\int_0^L L\cos\frac{n\pi x}{L}\,dx$$

$$= \int_0^L \cos\frac{n\pi x}{L}\,dx$$

$$= \begin{cases} \int_0^L 1\,dx = L & \text{if } n=0 \\ \frac{L}{n\pi}\sin\frac{n\pi x}{L}\Big|_0^L = 0 & \text{if } n=1,2,3\ldots \end{cases}$$

$$\underset{n=1,2,3\ldots}{b_n} = \frac{1}{L}\int_{-L}^{L} f(x)\sin\frac{n\pi x}{L}\,dx$$

$$= \frac{1}{L}\int_0^L L\sin\frac{n\pi x}{L}\,dx$$

$$= -\frac{L}{n\pi}\cos\frac{n\pi x}{L}\Big|_0^L$$

$$= -\frac{L}{n\pi}\cos(n\pi) + \frac{L}{n\pi}$$

$$= \begin{cases} -\dfrac{L}{n\pi} + \dfrac{L}{n\pi} = 0 & \text{if } n = \text{even} \\[2ex] -\dfrac{L}{n\pi}(-1) + \dfrac{L}{n\pi} = \dfrac{2L}{n\pi} & \text{if } n = \text{odd} \end{cases}$$

$$f(x) \text{ `` = '' } \frac{a_0}{2} + \sum_{n=1}^{\infty} \left(a_n \cos \frac{n\pi x}{L} + b_n \sin \frac{n\pi x}{L} \right)$$

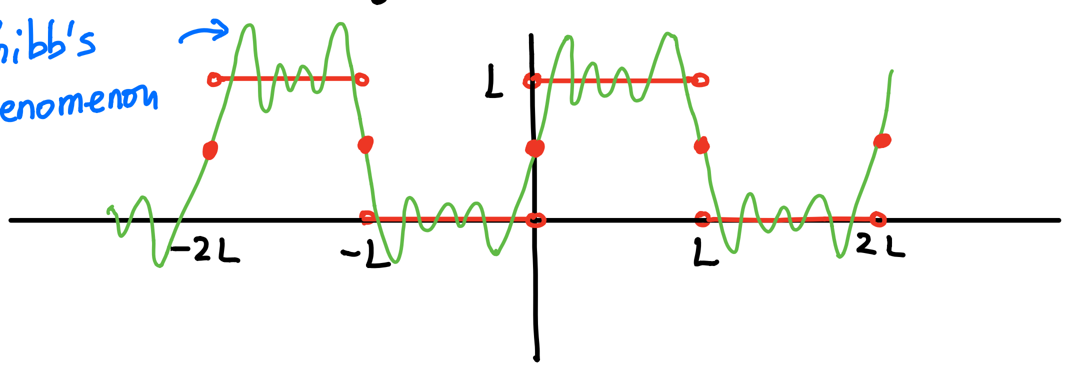
$$\text{`` = ''} \ \frac{L}{2} + \sum_{\substack{n=1 \\ n \text{ odd}}}^{\infty} \frac{2L}{n\pi} \sin \frac{n\pi x}{L}$$

$$\text{`` = ''} \ \frac{L}{2} + \frac{2L}{\pi} \sum_{m=1}^{\infty} \frac{1}{2m-1} \sin \left(\frac{(2m-1)\pi x}{L} \right)$$

Partial sum : $f_K(x) = \dfrac{L}{2} + \dfrac{2L}{\pi} \displaystyle\sum_{m=1}^{K} \dfrac{1}{2m-1} \sin \left(\dfrac{(2m-1)\pi x}{L} \right)$

Fourier series converges pointwise to the red function

Gibb's Phenomenon



partial sum

Convergence of Fourier Series

Notions of convergence $f_K \to f$

$$f_K(x) = \frac{a_0}{2} + \sum_{n=1}^{K} \left(a_n \cos\left(\frac{n\pi x}{L}\right) + b_n \sin\left(\frac{n\pi x}{L}\right)\right)$$

Pointwise convergence

For any fixed x_0, $|f_K(x_0) - f(x_0)| \to 0$ as $k \to \infty$.

Uniform convergence on $[-L, L]$

$$\max_{x \in [-L,L]} \left| f_K(x) - f(x) \right| \to 0 \quad \text{as } k \to \infty$$

Ex 3 does not converge uniformly

As $x \to 0^+$, $\begin{cases} f_k(x) \to \frac{L}{2}, \quad f(x) \to L \\ |f_k(x) - f(x)| \to \frac{L}{2} \text{ no matter how big } k \text{ is} \end{cases}$

L^2 convergence on $[-L, L]$

$$\| f_k(x) - f(x) \|^2 = \int_{-L}^{L} |f_k(x) - f(x)|^2 \, dx \to 0 \quad \text{as } k \to \infty$$

uniform $\Rightarrow$ pointwise and L^2

Fact: any $f \in L^2([-L,L])$ is the limit w.r.t. the L^2 norm of a sequence of continuous functions $f \in PS([-L,L])$.

Convergence theorems

Thm (Pointwise convergence) If $f \in PS([-L,L])$, then the Fourier series converges pointwise to $f(x)$ at all pts where $f(x)$ is continuous.

At a discontinuity x_0, it converges to the midpoint of the jump, i.e.

$$\frac{1}{2}\left(\lim_{x\to x_0^-} f(x) + \lim_{x\to x_0^+} f(x)\right).$$

Thm (Uniform convergence) If $f \in PS([-L,L])$ and f is continuous, then can change the word pointwise in the above statement to "uniformly".

Thm (L^2 convergence)

* If $f \in L^2([-L,L])$, then the Fourier series converges to $f(x)$ w.r.t. the L^2-norm.

* $\frac{a_0^2}{2} + \sum_{n=1}^{\infty}(a_n^2+b_n^2) = \frac{1}{L}\|f\|^2$ (Parseval's identity)

$\leq$ if only assumes $\frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\frac{n\pi x}{L} + b_n \sin\frac{n\pi x}{L}$ converges pointwise to f

* If $\{a_n\}_{n=0}^{\infty}$, $\{b_n\}_{n=1}^{\infty}$ are any sequence such that $\frac{a_0^2}{2} + \sum_{n=1}^{\infty} (a_n^2 + b_n^2)$ converges, then the series

$$\frac{a_0^2}{2} + \sum_{n=1}^{\infty} \left(a_n \cos\left(\frac{n\pi x}{L}\right) + b_n \sin\left(\frac{n\pi x}{L}\right)\right)$$ converges in L^2-norm to a function in $L^2([-L,L])$

might not be in $PS([-L,L])$