$x' = Ax$, A nondiagonalizable

<u>Ex 1</u>: $x' = Ax$, $A = \begin{bmatrix} 1 & -1 \\ 1 & 3 \end{bmatrix}$

<u>Eigenval</u>: $0 = \det(A - \lambda I) = (\lambda - 2)^2$

$\lambda = 2, 2$

<u>Eigenvec</u>: Solve $(A - 2I)v = 0$

$$\begin{bmatrix} -1 & -1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

i.e. $v_1 + v_2 = 0$

$$v = \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 1 \\ -1 \end{bmatrix} v_1 , \quad v_1 \in \mathbb{R} \setminus \{0\}$$

$\underbrace{\left\{ c_1 e^{2t} \begin{bmatrix} 1 \\ -1 \end{bmatrix} \right\}}_{\text{dim}=1}$ are solution, but there are more solutions that we need to find!

General soln to $x' = Ax$ is

$$x = e^{At} \begin{bmatrix} a \\ b \end{bmatrix} , \quad x(0) = \begin{bmatrix} a \\ b \end{bmatrix}$$

$$e^{At} = e^{2t} e^{(A-2I)t} \qquad \left(\text{because } e^{-2It} = e^{-2t} I \right)$$

$$= e^{2t} \left[I + (A-2I)t + \underbrace{\frac{(A-2I)^2 t^2}{2!} + \frac{(A-2I)^3 t^3}{3!} + \cdots}_{} \right]$$

$$= e^{2t}\left[I + (A-2I)t\right]$$

$$= e^{2t}\begin{bmatrix}1&0\\0&1\end{bmatrix} + te^{2t}\begin{bmatrix}-1&-1\\1&1\end{bmatrix}$$

= 0

due to Cayley-Hamilton thm

Let $p(\lambda) = \det(A-\lambda I)$

then $p(A) = 0$

In our ex:

$p(\lambda) = (\lambda-2)^2$

$p(A) = (A-2I)^2 = 0$

so $(A-2I)^k = 0$, $k \geq 2$.

Gen soln :

$$x(t) = \left(e^{2t}\begin{bmatrix}1&0\\0&1\end{bmatrix} + te^{2t}\begin{bmatrix}-1&-1\\1&1\end{bmatrix}\right)\begin{bmatrix}a\\b\end{bmatrix}$$

$$= e^{2t}\begin{bmatrix}a\\b\end{bmatrix} + te^{2t}\begin{bmatrix}-a-b\\a+b\end{bmatrix}$$

Valid soln, but want to write it in a way that is easier to graph, in a way where we see how the eigenvector v is involved.

v = eigenvector

* let u = any vector lin. indept from v, so u is not an eigenvector.

* General soln

$$x(t) = e^{At}\begin{bmatrix}a\\b\end{bmatrix} = a\,e^{At}\begin{bmatrix}1\\0\end{bmatrix} + b\,e^{At}\begin{bmatrix}0\\1\end{bmatrix}$$

$$\begin{bmatrix} a \\ b \end{bmatrix} = c_1 v + c_2 u \quad \text{for some } c_1, c_2$$

$$\Rightarrow x(t) = e^{At}(c_1 v + c_2 u)$$

$$x(t) = c_1 e^{At} v + c_2 e^{At} u$$

i.e. $x(t)$ is a linear combination of

$$e^{At} v = e^{2t}[I + (A-2I)t] v = e^{2t} v$$

and

$$e^{At} u = e^{2t}[I + (A-2I)t] u = e^{2t} u + t e^{2t}(A-2I) u$$

* Because $(A-2I)^2 u = 0$ b/c $(A-2I)^2 = 0$

$$(A-2I)\underbrace{(A-2I)\,u}_{w} = 0$$

$\Rightarrow (A-2I)u$ is an eigenvector, so

$(A-2I)u$ is proportional to v.

$$(A-2I)u = kv$$

$$(A-2I)\left(\frac{u}{k}\right) = v$$

can choose u s.t. $(A-2I)\,u = v$. u = generalized eigenvector

then $e^{At} u = e^{2t} u + t e^{2t} v$

Gen soln

$$\boxed{x(t) = c_1 e^{At} v + c_2 e^{At} u = c_1 e^{2t} v + c_2(e^{2t} u + t e^{2t} v)}$$

$$v = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$$

Solve $(A-2I)u = v$

$$\begin{bmatrix} -1 & -1 \\ 1 & 1 \end{bmatrix}\begin{bmatrix} u_1 \\ u_2 \end{bmatrix} = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$$

i.e. $\left.\begin{aligned} -u_1 - u_2 &= 1 \\ u_1 + u_2 &= -1 \end{aligned}\right\} \Rightarrow u_2 = -1 - u_1$

$$\begin{bmatrix} u_1 \\ u_2 \end{bmatrix} = \begin{bmatrix} u_1 \\ -1-u_1 \end{bmatrix} \neq 0$$

Pick one : $u = \begin{bmatrix} 0 \\ -1 \end{bmatrix}$

<u>Gen soln</u> : $\boxed{x(t) = c_1 e^{2t}\begin{bmatrix} 1 \\ -1 \end{bmatrix} + c_2\left(e^{2t}\begin{bmatrix} 0 \\ -1 \end{bmatrix} + te^{2t}\begin{bmatrix} 1 \\ -1 \end{bmatrix}\right)}$

$$= e^{2t}\begin{bmatrix} a \\ b \end{bmatrix} + te^{2t}\begin{bmatrix} -a-b \\ a+b \end{bmatrix}$$ when $a = c_1$, $-(a+b) = c_2$

<u>can check</u>

$$A = \begin{bmatrix} 1 & -1 \\ 1 & 3 \end{bmatrix} = P\begin{bmatrix} 2 & 1 \\ 0 & 2 \end{bmatrix}P^{-1}$$

$J =$ Jordan canonical form which is a generalization of diagonal matrices.

$$P = \begin{array}{cc} \begin{bmatrix} 1 & 0 \\ -1 & -1 \end{bmatrix} \\ \begin{array}{cc} v & u \end{array} \end{array}$$

$$x = c_1 e^{2t} v + c_2 e^{2t} u + c_2 t e^{2t} v$$

As $t \to \pm\infty$ $\begin{cases} \text{if } c_2 = 0, & x = c_1 e^{2t} v \\ \text{if } c_2 \neq 0, & c_2 t e^{2t} v \text{ dominates} \end{cases}$

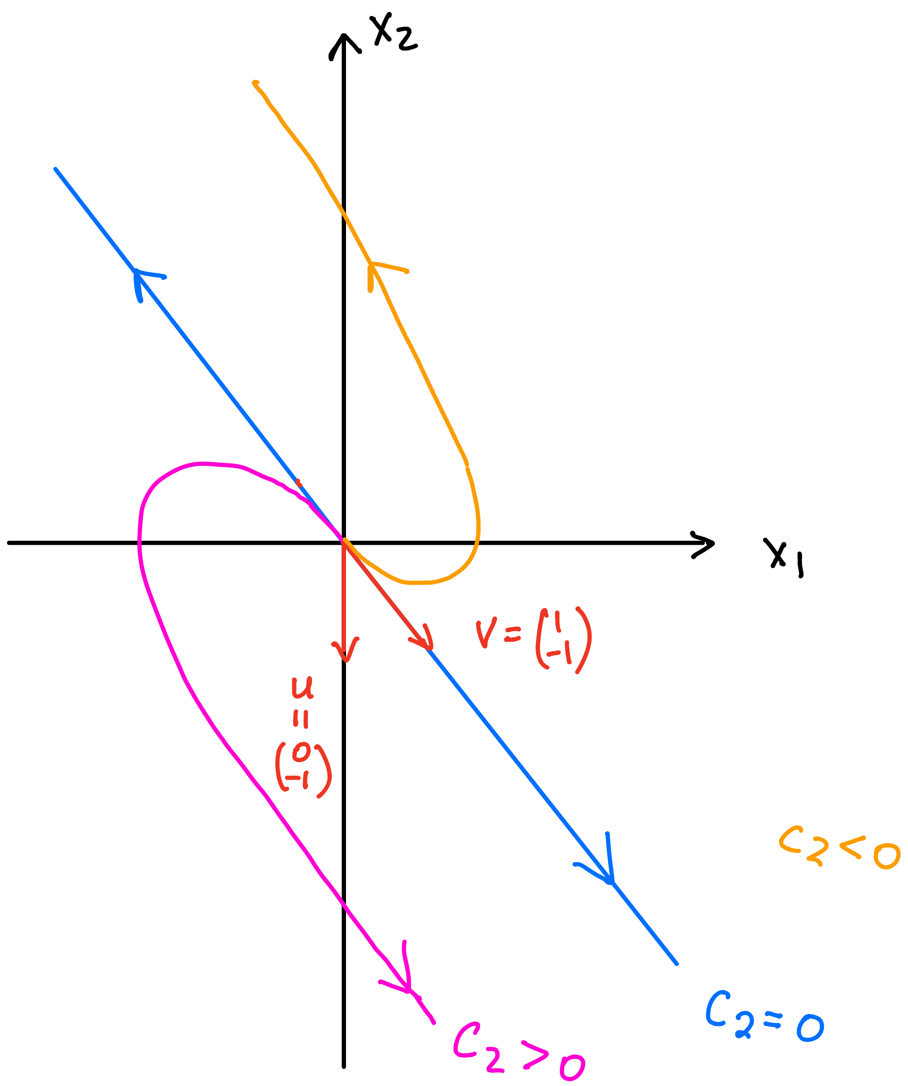
$x \to \infty$ when $t \to \infty$,

$x \to 0$ when $t \to -\infty$, so near the origin, $t < 0$ and $|t|$ big

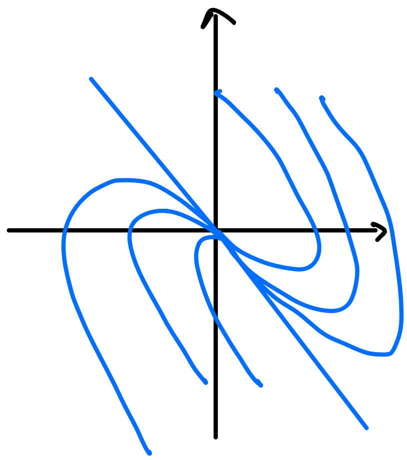
so $x = \underbrace{(c_1 + c_2 t)}_{\text{near the origin}} v e^{2t} + c_2 u e^{2t}$

$c_1 + c_2 t < 0$ if $c_2 > 0$

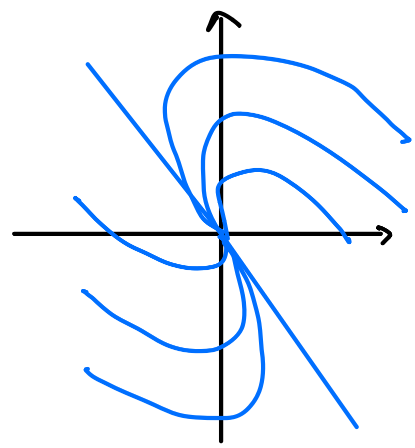
$c_1 + c_2 t > 0$ if $c_2 < 0$



In general, need to decide



VS.



Ex2 :

$$x' = Ax, \quad A = \begin{bmatrix} 1 & -1 \\ 1 & -1 \end{bmatrix}$$

Eigenval : $\lambda = 0$

Eigenvect : $v = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$

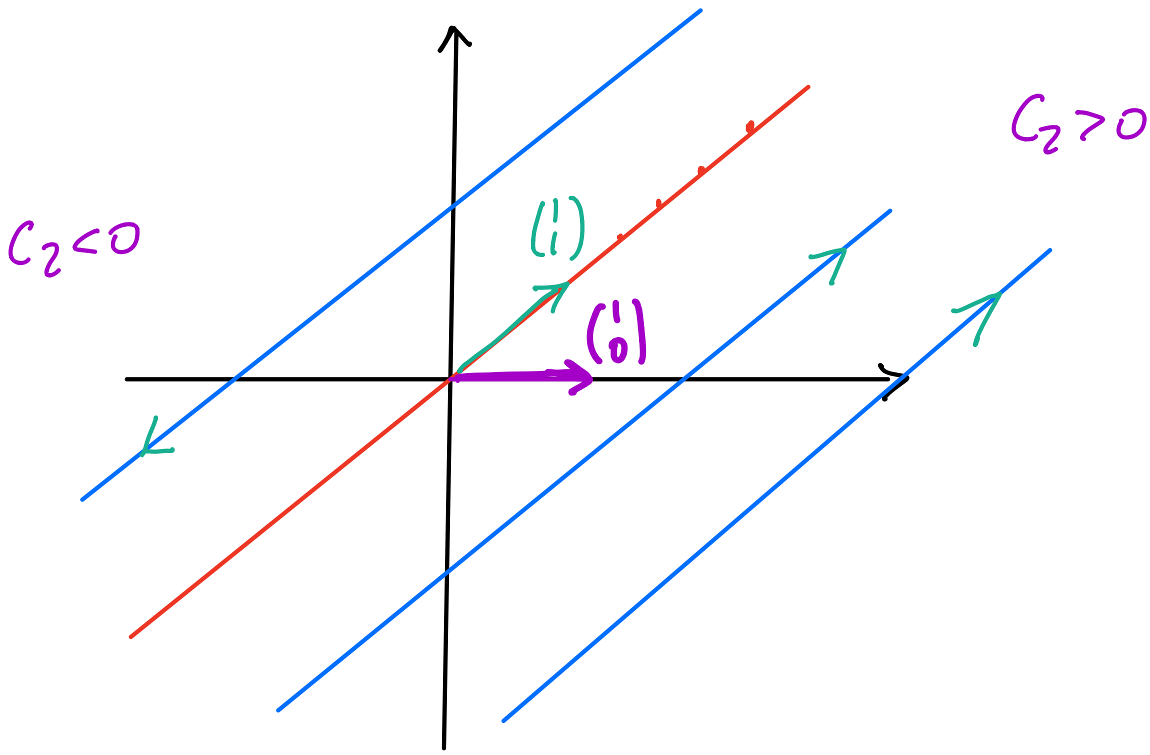
Generalized eigenvector : $(A - \lambda I)u = v$

$$\begin{bmatrix} 1 & -1 \\ 1 & -1 \end{bmatrix}\begin{bmatrix} u_1 \\ u_2 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

$$u = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

Gen soln

$$x = c_1 \begin{bmatrix} 1 \\ 1 \end{bmatrix} + c_2 \left[\begin{pmatrix} 1 \\ 0 \end{pmatrix} + t \begin{pmatrix} 1 \\ 1 \end{pmatrix} \right]$$


$C_2 < 0$
$\begin{pmatrix}1\\1\end{pmatrix}$
$\begin{pmatrix}1\\0\end{pmatrix}$
$C_2 > 0$