$x' = Ax$, A 2×2 complex eigenvalues

<u>Ex1</u> $x' = \underbrace{\begin{bmatrix} -1 & -2 \\ 8 & -1 \end{bmatrix}}_{A} x$

<u>Eigenval</u>: $0 = \det \begin{bmatrix} -1-\lambda & -2 \\ 8 & -1-\lambda \end{bmatrix} = (-1-\lambda)^2 + 16 = \lambda^2 + 2\lambda + 17$

$$\lambda = \frac{-2 \pm \sqrt{4-68}}{2} = -1 \pm 4i$$

<u>Fact</u>: If A is real, v an eigenvec. w/ eigenval λ

$\Leftrightarrow$ $\bar{v}$ is an eigenvector w/ eigenvalue $\bar{\lambda}$.

reason: $Av = \lambda v$

$\Leftrightarrow$ $A\bar{v} = \overline{Av} = \overline{\lambda v} = \bar{\lambda}\bar{v}$

Eigenvector for $\boxed{\lambda = -1 + 4i}$ (pick one)

solve $(A - \lambda I)v = 0$

$$\begin{bmatrix} -4i & -2 \\ 8 & -4i \end{bmatrix}\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \rightsquigarrow -4iv_1 - 2v_2 = 0$$

$\Rightarrow v_2 = -2iv_1$

$$v = \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} v_1 \\ -2iv_2 \end{bmatrix} = \begin{bmatrix} 1 \\ -2i \end{bmatrix} v_1, \quad v_1 \neq 0$$

Pick one: $\boxed{v = \begin{bmatrix} 1 \\ -2i \end{bmatrix}}$

$$\underline{\text{Gen soln}}: \quad x = \underbrace{c_1 e^{(-1+4i)t}\begin{bmatrix}1\\-2i\end{bmatrix}}_{y=u+iv} + \underbrace{c_2 e^{(-1-4i)t}\begin{bmatrix}1\\2i\end{bmatrix}}_{\bar{y}=u-iv}$$

complex basis of soln's

But we want to write the soln's in terms of real valued functions instead (physically more meaningful)

<u>Note</u>: Soln to $x' = Ax$ is $x = e^{At}x(0)$, so if A is real and $x(0)$ is real, then x should be real.

Euler's formula

$$e^{i\theta} = \cos\theta + i\sin\theta$$

$$e^{a+i\theta} = e^a\cos\theta + ie^a\sin\theta$$

$$y = e^{(-1+4i)t}\begin{bmatrix}1\\-2i\end{bmatrix}$$

$$= e^{-t+i(4t)}\begin{bmatrix}1\\-2i\end{bmatrix}$$

$$= \left(e^{-t}\cos(4t) + ie^{-t}\sin(4t)\right)\left(\begin{bmatrix}1\\0\end{bmatrix} + i\begin{bmatrix}0\\-2\end{bmatrix}\right)$$

$$= \underbrace{\left(e^{-t}\cos 4t\begin{bmatrix}1\\0\end{bmatrix} - e^{-t}\sin 4t\begin{bmatrix}0\\-2\end{bmatrix}\right)}_{\text{Re } y = u} + i\underbrace{\left(e^{-t}\sin 4t\begin{bmatrix}1\\0\end{bmatrix} + e^{-t}\cos 4t\begin{bmatrix}0\\-2\end{bmatrix}\right)}_{\text{Im } y = v}$$

Both u and v are soln's to $x' = Ax$

Reason 1: $y = u + iv$, $y' = Ay$

$$(u+iv)' = A(u+iv)$$

$$u' + iv' = Au + iAv$$

$$\Rightarrow \begin{cases} u' = Au \\ v' = Av \end{cases}$$

Reason 2

$$\left.\begin{aligned} y &= u + iv \\ \bar{y} &= u - iv \end{aligned}\right\} \xleftrightarrow[\text{basis}]{\text{Change of}} \begin{cases} u = \dfrac{y+\bar{y}}{2} = \operatorname{Re} y \\[2ex] v = \dfrac{y-\bar{y}}{2i} = \operatorname{Im} y \end{cases}$$

Gen soln to $x' = Ax$ in terms of real basis is

$$\boxed{x = c_1 \operatorname{Re} y + c_2 \operatorname{Im} y}$$

$$X = c_1\left(e^{-t}\cos 4t \begin{bmatrix} 1 \\ 0 \end{bmatrix} - e^{-t}\sin 4t \begin{bmatrix} 0 \\ -2 \end{bmatrix}\right) + c_2\left(e^{-t}\sin 4t \begin{bmatrix} 1 \\ 0 \end{bmatrix} + e^{-t}\cos 4t \begin{bmatrix} 0 \\ -2 \end{bmatrix}\right)$$

$$= c_1 \begin{bmatrix} e^{-t}\cos 4t \\ 2e^{-t}\sin 4t \end{bmatrix} + c_2 \begin{bmatrix} e^{-t}\sin 4t \\ -2e^{-t}\cos 4t \end{bmatrix}$$

As $t \to \infty$, $x(t) \to 0$

$$x(t) = e^{-t}\begin{bmatrix} \cos 4t & \sin 4t \\ 2\sin 4t & -2\cos 4t \end{bmatrix}\begin{bmatrix} c_1 \\ c_2 \end{bmatrix}$$

$$= \underbrace{e^{-t}\begin{bmatrix} \cos 4t & -\frac{\sin 4t}{2} \\ 2\sin 4t & \cos 4t \end{bmatrix}}_{e^{At}} \begin{bmatrix} C_1 \\ -2C_2 \end{bmatrix}$$

$$\boxed{X(0) = \begin{bmatrix} 1 & 0 \\ 0 & -2 \end{bmatrix}\begin{bmatrix} C_1 \\ C_2 \end{bmatrix} = \begin{bmatrix} C_1 \\ -2C_2 \end{bmatrix}}$$

$$= e^{At} X(0)$$

Ex2 : $A = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$, $x' = Ax$

Eigenval : $\pm i$, pick $\boxed{\lambda = i}$

Corresp. eigenvec : $\boxed{V = \begin{bmatrix} 1 \\ i \end{bmatrix}}$

$$y = e^{it}\begin{bmatrix} 1 \\ i \end{bmatrix} = (\cos t + i\sin t)\left(\begin{bmatrix} 1 \\ 0 \end{bmatrix} + i\begin{bmatrix} 0 \\ 1 \end{bmatrix}\right)$$

$$X(t) = C_1 \operatorname{Re}(y) + C_2 \operatorname{Im}(y)$$

$$= C_1\left(\cos t\begin{bmatrix} 1 \\ 0 \end{bmatrix} - \sin t\begin{bmatrix} 0 \\ 1 \end{bmatrix}\right) + C_2\left(\sin t\begin{bmatrix} 1 \\ 0 \end{bmatrix} + \cos t\begin{bmatrix} 0 \\ 1 \end{bmatrix}\right)$$

$$= C_1\begin{bmatrix} \cos t \\ -\sin t \end{bmatrix} + C_2\begin{bmatrix} \sin t \\ \cos t \end{bmatrix}$$

$$x(t) = \underbrace{\begin{bmatrix}\cos t & \sin t\\ -\sin t & \cos t\end{bmatrix}}_{e^{At}}\begin{bmatrix}c_1\\ c_2\end{bmatrix}$$

$$x(0) = \begin{bmatrix}1 & 0\\ 0 & 1\end{bmatrix}\begin{bmatrix}c_1\\ c_2\end{bmatrix} = \begin{bmatrix}c_1\\ c_2\end{bmatrix}$$

$$= e^{At}\, x(0)$$

__RMK:__ $e^{At} = P e^{Dt} P^{-1} = P\begin{bmatrix}e^{it} & 0\\ 0 & e^{-it}\end{bmatrix}P^{-1} = \begin{bmatrix}\cos t & \sin t\\ -\sin t & \cos t\end{bmatrix}$

HW4 #2

$A = PDP^{-1}$

$P = \begin{bmatrix}1 & 1\\ i & -i\end{bmatrix}$, $D = \begin{bmatrix}i & 0\\ 0 & -i\end{bmatrix}$

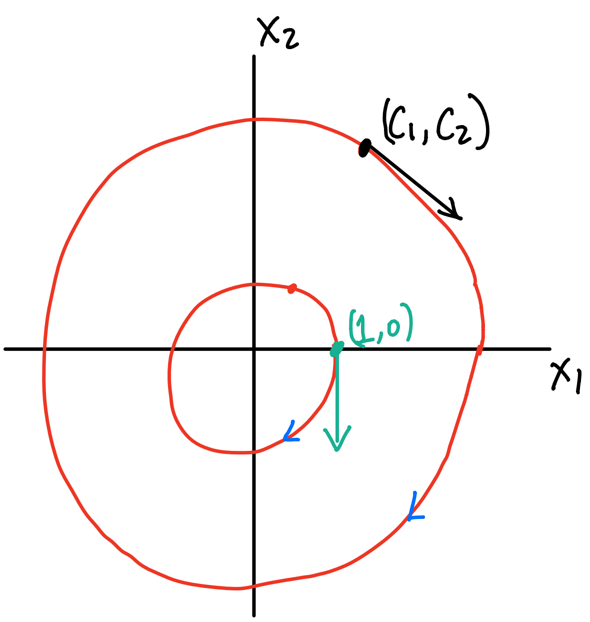
Ex 2 cont'd

$$x_1 = c_1\cos t + c_2 \sin t$$

$$x_2 = -c_1 \sin t + c_2 \cos t$$

__can Verify__ : $x_1^2 + x_2^2 = c_1^2 + c_2^2$

Circle centered at the __origin__ of radius $\sqrt{c_1^2+c_2^2}$



$$x(0) = \begin{pmatrix}c_1\\ c_2\end{pmatrix}$$

$$x'(0) = A\,x(0) = \begin{bmatrix}0 & 1\\ -1 & 0\end{bmatrix}\begin{bmatrix}c_1\\ c_2\end{bmatrix} = \begin{bmatrix}c_2\\ -c_1\end{bmatrix}$$

* all circles go in the same direction, so can use any "easy" initial condition to check the direction

Easy test case: when $x(0) = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$

$$x'(0) = A\,x(0) = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ -1 \end{bmatrix}$$

Ex 3 sketch $x = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} \cos 4t & \sin 4t \\ 2\sin 4t & -2\cos 4t \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \end{bmatrix}$

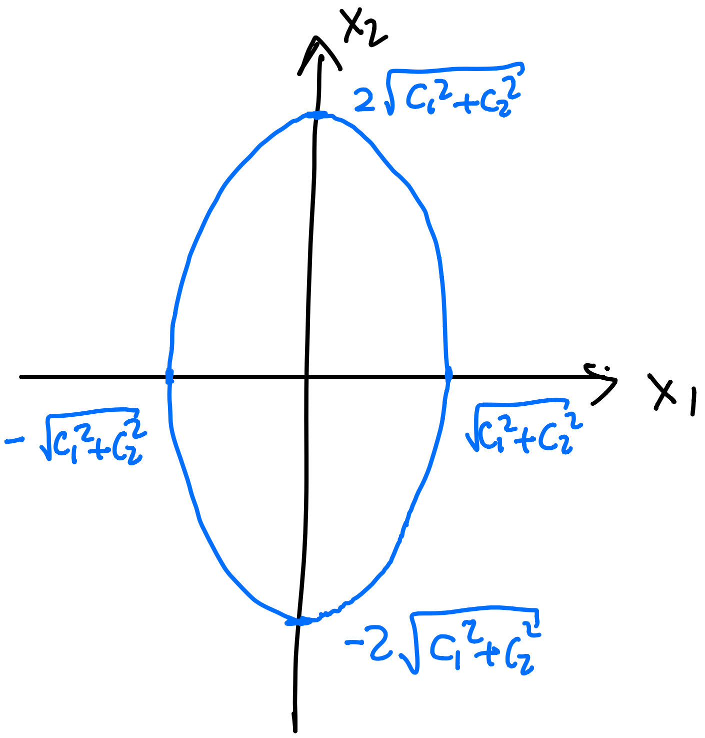
$$x_1 = c_1 \cos 4t + c_2 \sin 4t$$

$$x_2 = 2\left(c_1 \sin 4t - c_2 \cos 4t\right)$$

can verify $x_1^2 + \left(\frac{x_2}{2}\right)^2 = c_1^2 + c_2^2$

$\Rightarrow$ ellipse centered at the origin

$\Rightarrow$ axis of the ellipse = x_1, x_2 axes



$$\left(\frac{x_1}{a}\right)^2 + \left(\frac{x_2}{b}\right)^2 = c \qquad \text{has "straight" axis}$$

Rmk: HW 5 # 1 find axes for ellipse with tilted axis.

In general:

If $\lambda = ib$ (purely imaginary)

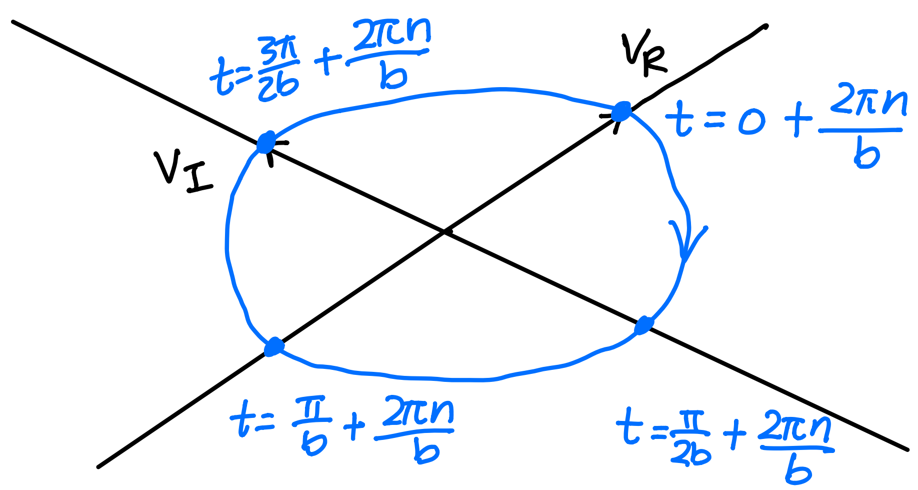
$$V = V_R + i V_I$$

One real soln is

$$\mathrm{Re}\left(e^{ibt} v\right)$$

$$= \mathrm{Re}\left((\cos bt + i \sin bt)(V_R + i V_I)\right)$$

$$= \cos(bt)\, V_R - (\sin bt)\, V_I$$



$, n \in \mathbb{Z}$

* But V_I and V_R may not be axes of the ellipse,

not when V_I and V_R are not $\perp$.

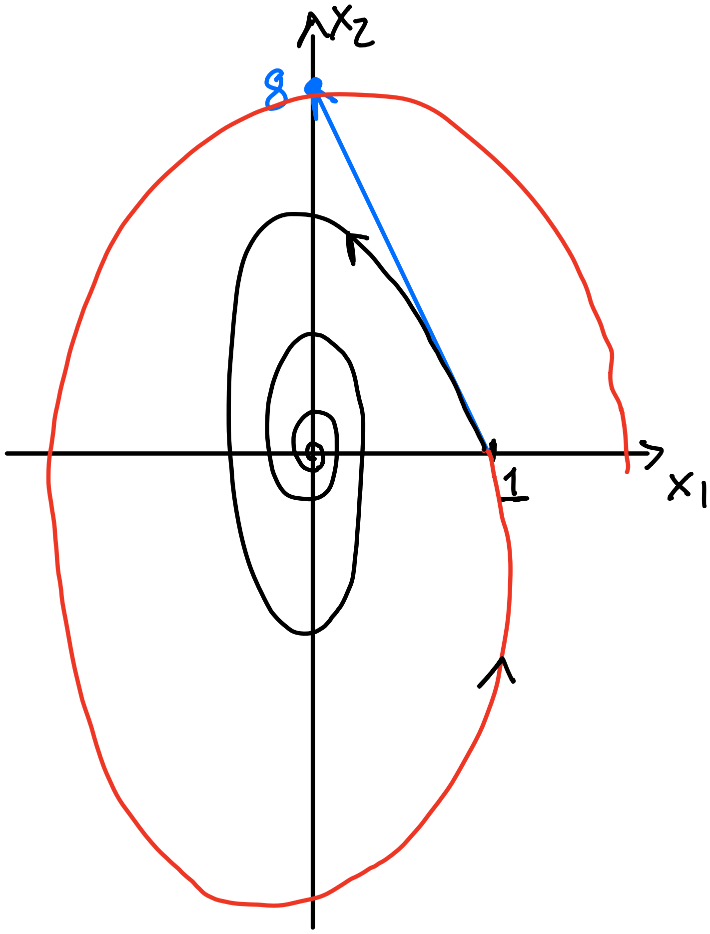
* other trajectories are concentric ellipses

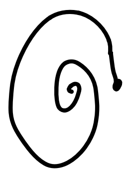
Ex 1 (cont'd) eigenvalue $\lambda = a \pm bi$, $a \neq 0$

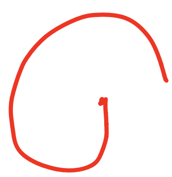
$$x(t) = e^{-t}\begin{bmatrix} \cos 4t & \sin 4t \\ 2\sin 4t & -2\cos 4t \end{bmatrix}\begin{bmatrix} c_1 \\ c_2 \end{bmatrix}$$ elliptical spiral

$x' = Ax$, $A = \begin{bmatrix} -1 & -2 \\ 8 & -1 \end{bmatrix}$, $x(0) = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$

$$x'(0) = A\,x(0)$$
$$= \begin{bmatrix} -1 & -2 \\ 8 & -1 \end{bmatrix}\begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} -1 \\ 8 \end{bmatrix}$$



 $t \in [0, \infty)$

 $t \leq 0$

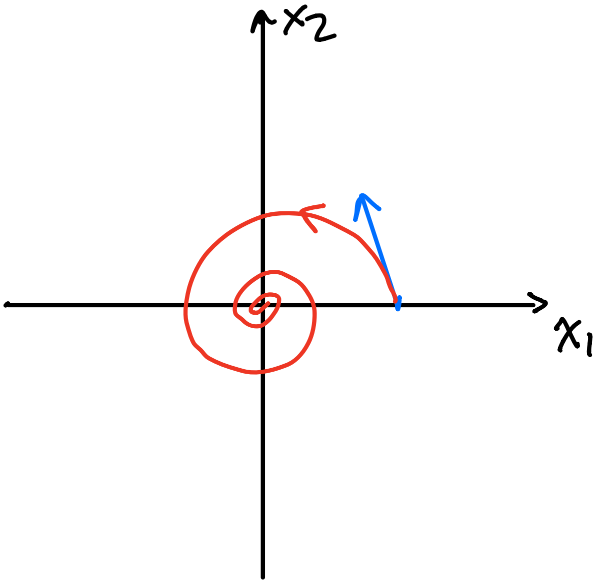
Stable spiral

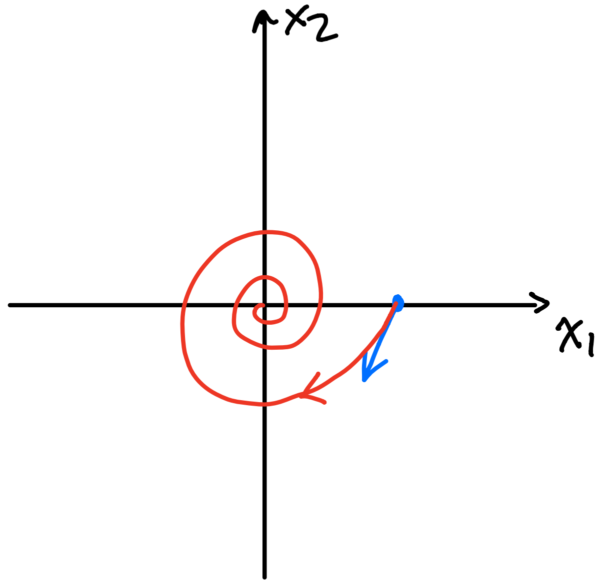
counterclockwise

Different spiral

Stable Spiral

trajectory for $t \geq 0$





unstable Spiral