Homogeneous 1st order system with constant coefficient

1st order linear eqn : $x' = P(t)x + g(t)$, $x = \begin{bmatrix} x_1(t) \\ \vdots \\ x_n(t) \end{bmatrix}$

homogeneous linear eqn : $x' = P(t)x$

homogeneous with constant coefficient : $x' = Ax$

$\uparrow$ $n \times n$ constant entries

Ex1 $x'(t) = \begin{bmatrix} -1 & 0 \\ 0 & -3 \end{bmatrix} x(t)$

i.e. $\left.\begin{aligned} x_1' &= -x_1 \\ x_2' &= -3x_2 \end{aligned}\right\} \Rightarrow \begin{aligned} x_1 &= c_1 e^{-t} \\ x_2 &= c_2 e^{-3t} \end{aligned}$

$\Rightarrow x(t) = \begin{bmatrix} c_1 e^{-t} \\ c_2 e^{-3t} \end{bmatrix} = c_1 \begin{bmatrix} 1 \\ 0 \end{bmatrix} e^{-t} + c_2 \begin{bmatrix} 0 \\ 1 \end{bmatrix} e^{-3t}$

Observe : eigenval of $\begin{bmatrix} -1 & 0 \\ 0 & -3 \end{bmatrix}$ are $\lambda_1 = -1$, $\lambda_2 = -3$

corresp. eigenvec : $v_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$, $v_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$

Ex 2 $x'(t) = \begin{bmatrix} 2 & -1 \\ -1 & 2 \end{bmatrix} x(t)$

In worksheet, we found

Eigenval: $\lambda_1 = 1, \quad \lambda_2 = 3$

Corresp eigenvec: $v_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \quad v_2 = \begin{bmatrix} -1 \\ 1 \end{bmatrix}$ (chose one)

$$\begin{bmatrix} 2 & -1 \\ -1 & 2 \end{bmatrix} = P \begin{bmatrix} 1 & 0 \\ 0 & 3 \end{bmatrix} P^{-1}, \quad P = \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix}$$

$$x' = P \begin{bmatrix} 1 & 0 \\ 0 & 3 \end{bmatrix} P^{-1} x$$

$$(P^{-1}x)' = P^{-1}x' = \begin{bmatrix} 1 & 0 \\ 0 & 3 \end{bmatrix} (P^{-1}x)$$

$$y = P^{-1}x$$

$$y' = \begin{bmatrix} 1 & 0 \\ 0 & 3 \end{bmatrix} y \quad \text{, i.e. } \begin{array}{l} y_1' = y_1 \\ y_2' = 3y_2 \end{array}$$

$$y = \begin{bmatrix} c_1 e^t \\ c_2 e^{3t} \end{bmatrix}$$

$$y = c_1 \begin{bmatrix} 1 \\ 0 \end{bmatrix} e^t + c_2 \begin{bmatrix} 0 \\ 1 \end{bmatrix} e^{3t}$$

$$x = Py = \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix} \left(c_1 \begin{bmatrix} 1 \\ 0 \end{bmatrix} e^t + c_2 \begin{bmatrix} 0 \\ 1 \end{bmatrix} e^{3t} \right)$$

$$x = c_1 e^t \begin{bmatrix} 1 \\ 1 \end{bmatrix} + c_2 e^{3t} \begin{bmatrix} -1 \\ 1 \end{bmatrix}$$

<u>key observation</u>: for $x' = Ax$

> If v is an eigenvector of A with eigenvalue λ,
> then $x = e^{\lambda t} v$ is a solution.

Indeed: $x' = \lambda e^{\lambda t} v$

$$Ax = A(e^{\lambda t} v) = e^{\lambda t} A v = e^{\lambda t}(\lambda v) = \lambda e^{\lambda t} v = x'$$

Fact: A (n×n) diagonalizable $\Leftrightarrow$ A has n linearly indept. eigenvectors $v_1, \dots, v_n$

i.e. $A = PDP^{-1}$

by key obser. $\Longrightarrow$ Suppose $\lambda_1, \dots, \lambda_n$ are the corresp. eigenvalues then each $e^{\lambda_j t} v_j$, $j = 1, \dots, n$ is a solution, (This is a set of n linearly independent solutions)

Fact (HW3, #1): For homogeneous linear eqn's, i.e. $x' = P(t)x$, if $x^{(1)}$ and $x^{(2)}$ are solutions, then

$x = c_1 x^{(1)} + c_2 x^{(2)}$ is also a soln.

Gen soln to $x' = Ax$ when A is diagonalizable:

$$\boxed{x = c_1 e^{\lambda_1 t} v_1 + \cdots + c_n e^{\lambda_n t} v_n}$$

(Note: we didn't assume λ and v real, so this works for complex ones also).

Ex3 $x' = \begin{bmatrix} 1 & 1 \\ 4 & 1 \end{bmatrix} x$

eigenval: $\lambda_1 = 3$, $\lambda_2 = -1$

Corresp. eigenvec: $v_1 = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$, $v_2 = \begin{bmatrix} 1 \\ -2 \end{bmatrix}$

$\Rightarrow$ $x = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = c_1 e^{3t} \begin{bmatrix} 1 \\ 2 \end{bmatrix} + c_2 e^{-t} \begin{bmatrix} 1 \\ -2 \end{bmatrix}$.

For $n\times n$ matrix: If $v_1, \dots, v_n$ are eigenvectors with with eigenvalues $\lambda_1, \dots, \lambda_n$, respectively and such that $\lambda_1, \dots, \lambda_n$ are distinct,

then $v_1, \dots, v_n$ are linearly independant.

(Same statement is true if using $v_1, \dots, v_m$ and $\lambda_1, \dots, \lambda_m$ for $m \le n$)

For 2×2 matrix A 2 eigenval λ_1, λ_2

① If $\lambda_1 \neq \lambda_2$, then A is diagonalizable

② If $\lambda_1 = \lambda_2 = \lambda$, then

$$A \text{ diagonalizable} \overset{\text{HW1 \#6}}{\Longleftrightarrow} A = \begin{bmatrix} \lambda & 0 \\ 0 & \lambda \end{bmatrix}$$

Ex 4 $\quad x' = Ax, \quad A = \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix}$

eigenval: $\lambda = 2, 2$

eigenvec: Solve $(A - 2I)v = 0$

i.e. $\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$

$\begin{bmatrix} v_1 \\ v_2 \end{bmatrix}$ can be anything

$v^A = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \quad v^B = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$ or any two linearly indep. vectors

$\Rightarrow$ Gen soln: $x(t) = c_1 e^{2t} \begin{bmatrix} 1 \\ 0 \end{bmatrix} + c_2 e^{2t} \begin{bmatrix} 0 \\ 1 \end{bmatrix}$

Another Perspective

$$x' = Ax, \qquad x = \begin{bmatrix} x_1(t) \\ \vdots \\ x_n(t) \end{bmatrix}$$

Gen soln: $\left\{ \text{recall single var: } x' = ax, \ x \text{ scalar} \right.$

$\Rightarrow x = Ce^{at}.$

$$x = e^{At} c, \qquad c = \begin{bmatrix} c_1 \\ \vdots \\ c_n \end{bmatrix} \text{ arbitrary constants}$$

$$\boxed{\text{Def}^{\underline{n}} \text{ of exp}: \ \frac{d}{dt}(e^{At}) = Ae^{At} \text{ and } e^{[0]} = I_{n\times n}}$$

check: $x' = \frac{d}{dt}(e^{At}c) = \frac{d}{dt}(e^{At})c = A(e^{At}c) = Ax$

Initial contions: Given $x(0) = \begin{bmatrix} x_1(0) \\ \vdots \\ x_n(0) \end{bmatrix}$, can determine c

$$\begin{bmatrix} x_1(0) \\ \vdots \\ x_n(0) \end{bmatrix} = x(0) = \underbrace{e^{A0}}_{I_{n\times n}} c = c$$

Matrix exponential $\quad f(t) = e^{At}$

Taylor $f(t) = f(0) + f'(0)t + \frac{f''(0)t^2}{2!} + \frac{f'''(0)t^3}{3!} + \cdots$

$f(0) = A, \quad f''(0) = A^2, \quad f''(0) = A^3$

$$e^{At} = I + At + \frac{A^2t^2}{2!} + \frac{A^3t^3}{3!} + \cdots$$

Everything below assumes A diagonalizable, $A = PDP^{-1}$

then $A^2 = (PDP^{-1})(PDP^{-1}) = PD^2P^{-1}$

$$A = PD^{K}P^{-1}$$

$$e^{At} = \underset{\substack{\uparrow \\ PIP^{-1}}}{I} + PDP^{-1}t + \frac{PD^2P^{-1}t^2}{2!} + \frac{PD^3P^{-1}t^3}{3!} + \cdots$$

$$e^{At} = P\left(I + Dt + \frac{D^2t^2}{2!} + \frac{D^3t^3}{3!} + \cdots\right)P^{-1}$$

$$e^{At} = Pe^{Dt}P^{-1}$$

Computing e^{Dt}

$$e^{Dt} = I + Dt + \frac{D^2t^2}{2!} + \frac{D^3t^3}{3!} + \cdots$$

$$= \begin{bmatrix} 1 + \lambda_1 t + \frac{\lambda_1^2 t^2}{2!} + \cdots & & 0 \\ & \ddots & \\ 0 & & 1 + \lambda_n t + \frac{\lambda_n^2 t^2}{2!} + \cdots \end{bmatrix}$$

$$e^{Dt} = \begin{bmatrix} e^{\lambda_1 t} & & 0 \\ & \ddots & \\ 0 & & e^{\lambda_n t} \end{bmatrix}$$

$$D = \begin{bmatrix} \lambda_1 & & 0 \\ & \ddots & \\ 0 & & \lambda_n \end{bmatrix}, \text{ so } D^k = \begin{bmatrix} \lambda_1^k & & 0 \\ & \ddots & \\ 0 & & \lambda_n^k \end{bmatrix}.$$

$$e^{At} = Pe^{Dt}P^{-1} = P \begin{bmatrix} e^{\lambda_1 t} & & 0 \\ & \ddots & \\ 0 & & e^{\lambda_n t} \end{bmatrix} P^{-1}$$

$$x = e^{At} \begin{bmatrix} c_1 \\ \vdots \\ c_n \end{bmatrix} = P \begin{bmatrix} e^{\lambda_1 t} & & 0 \\ & \ddots & \\ 0 & & e^{\lambda_n t} \end{bmatrix} \begin{bmatrix} d_1 \\ \vdots \\ d_n \end{bmatrix}, \quad \begin{bmatrix} d_1 \\ \vdots \\ d_n \end{bmatrix} = P^{-1} \begin{bmatrix} c_1 \\ \vdots \\ c_n \end{bmatrix}$$

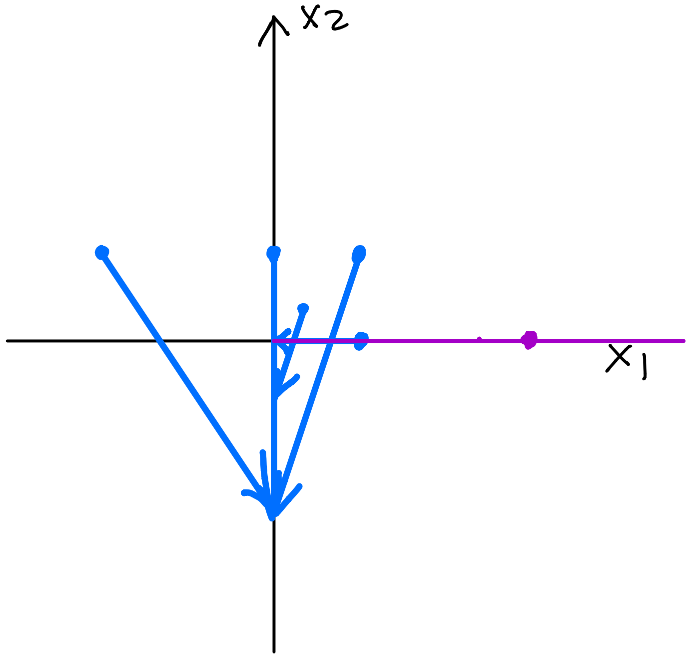
$$= P \left(d_1 e^{\lambda_1 t} \begin{bmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix} + d_2 e^{\lambda_2 t} \begin{bmatrix} 0 \\ 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix} + \cdots + d_n e^{\lambda_n t} \begin{bmatrix} 0 \\ \vdots \\ 0 \\ 1 \end{bmatrix} \right)$$

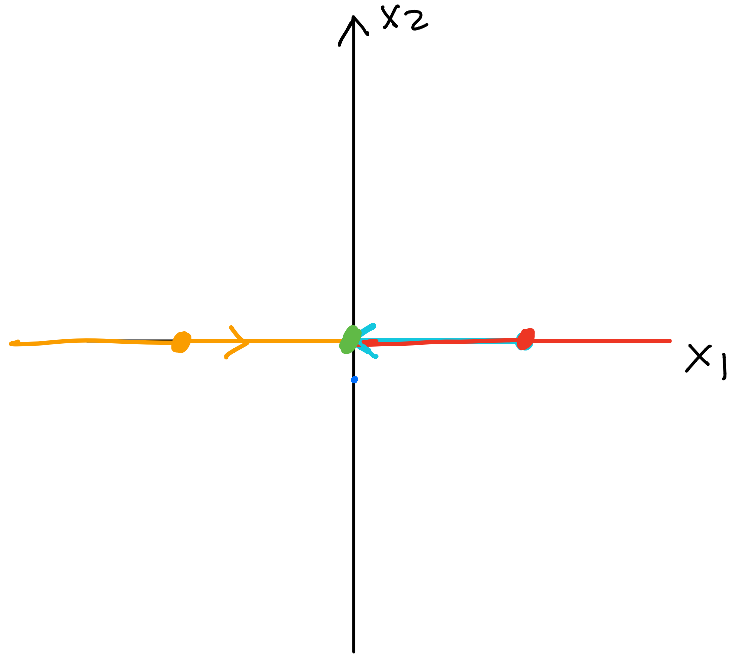
$$x = d_1 e^{\lambda_1 t} v_1 + d_2 e^{\lambda_2 t} v_2 + \cdots + d_n e^{\lambda_n t} v_n$$

Phase portrait

Ex 1 (Cont'd) $A = \begin{bmatrix} -1 & 0 \\ 0 & -3 \end{bmatrix}$, $x' = Ax$

Direction field: for any x, $Ax = x'$ gives us a "direction" in the x_1-x_2 plane





In class exercise :

① For pts $x = \begin{pmatrix}1\\0\end{pmatrix}, \begin{pmatrix}2\\0\end{pmatrix}, \begin{pmatrix}0\\1\end{pmatrix}, \begin{pmatrix}1\\1\end{pmatrix}, \begin{pmatrix}\frac{1}{3}\\\frac{1}{3}\end{pmatrix}, \begin{pmatrix}-2\\1\end{pmatrix}$, draw x'.

$x' = \begin{pmatrix}-1\\0\end{pmatrix}, \begin{pmatrix}-2\\0\end{pmatrix}, \begin{pmatrix}0\\-3\end{pmatrix}, \begin{pmatrix}-1\\-3\end{pmatrix}, \begin{pmatrix}-\frac{1}{3}\\-1\end{pmatrix}, \begin{pmatrix}2\\-3\end{pmatrix}$

② we know $x(t) = c_1 \begin{bmatrix}1\\0\end{bmatrix} e^{-t} + c_2 \begin{bmatrix}0\\1\end{bmatrix} e^{-3t} = \begin{bmatrix}c_1 e^{-t}\\c_2 e^{-3t}\end{bmatrix}$

Suppose $x(0) = \begin{pmatrix}2\\0\end{pmatrix}$, what is $x(t)$? Draw the trajector for $t \in (-\infty, \infty)$

$\begin{pmatrix}2\\0\end{pmatrix} = x(0) = \begin{pmatrix}c_1\\c_2\end{pmatrix}$

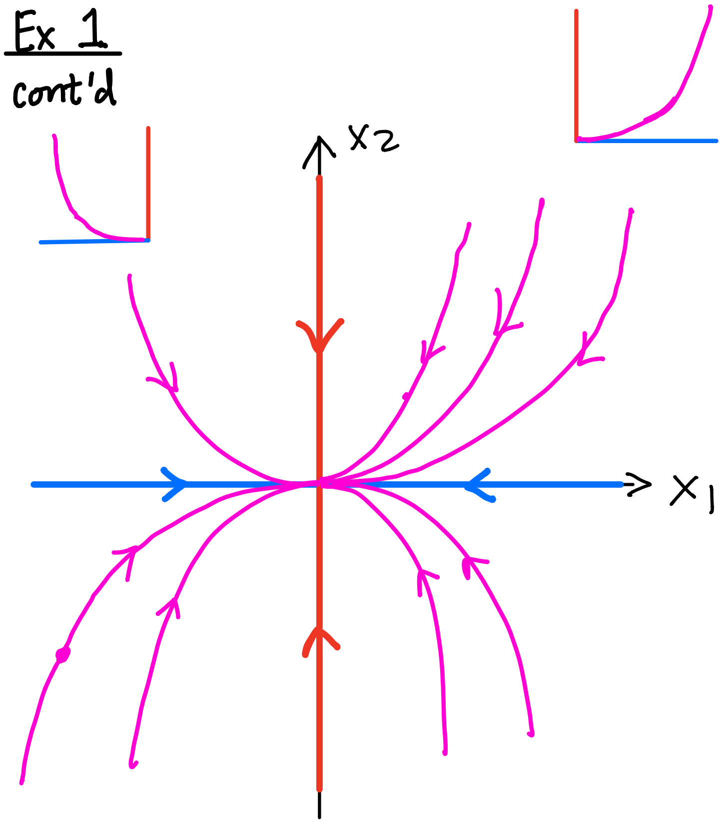
$x(t) = \begin{bmatrix}2e^{-t}\\0\end{bmatrix} = \begin{bmatrix}2\\0\end{bmatrix} e^{-t}$

③ What about when $x(0) = \begin{pmatrix}3\\0\end{pmatrix}$? $x(t) = \begin{bmatrix}3\\0\end{bmatrix} e^{-t}$

④ What about when $x(0) = \begin{pmatrix}-2\\0\end{pmatrix}$? $x'(0) = \begin{pmatrix}2\\0\end{pmatrix}$
$x(t) = \begin{bmatrix}-2\\0\end{bmatrix} e^{-t}$

⑤ What about when $x(0) = \begin{pmatrix}0\\0\end{pmatrix}$? $x'(0) = \begin{pmatrix}0\\0\end{pmatrix}$
$x(t) = \begin{pmatrix}0\\0\end{pmatrix}$

Ex 1
cont'd



$$x' = \begin{bmatrix} -1 & 0 \\ 0 & -3 \end{bmatrix} x$$

$$x = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} c_1 e^{-t} \\ c_2 e^{-3t} \end{pmatrix}$$

	$c_1 \begin{bmatrix} 1 \\ 0 \end{bmatrix} e^{-t}$	$c_2 \begin{bmatrix} 0 \\ 1 \end{bmatrix} e^{-3t}$
as $t \to \infty$	$\to 0$ dominates	$\to 0$
$t \to -\infty$	$\to \infty$	$-\infty$ dominates

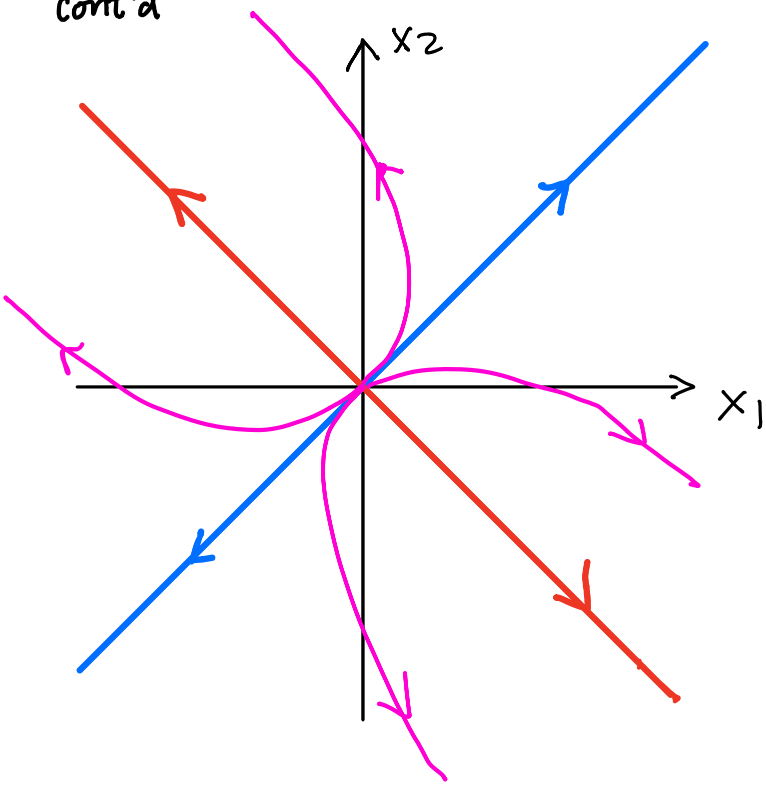
always opposite of $t \to \infty$

All soln's $x(t) \to 0$ as $t \to \infty$

$x = 0$ is an asymtotically stable equilibrium

uniquess of soln of linear eqn's
$\Rightarrow$ trajectries don't cross

Ex 2
cont'd



$$x' = \begin{bmatrix} 2 & -1 \\ -1 & 2 \end{bmatrix} x$$

$$x = \begin{pmatrix} c_1 e^{t} - c_2 e^{3t} \\ c_1 e^{t} + c_2 e^{3t} \end{pmatrix}$$

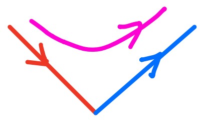
	$c_1 \begin{bmatrix} 1 \\ 1 \end{bmatrix} e^{t}$	$c_2 \begin{bmatrix} -1 \\ 1 \end{bmatrix} e^{3t}$
$t \to \infty$	$\to \infty$	$\to \infty$ dominate
$t \to -\infty$	$\to 0$ dominate	$\to 0$

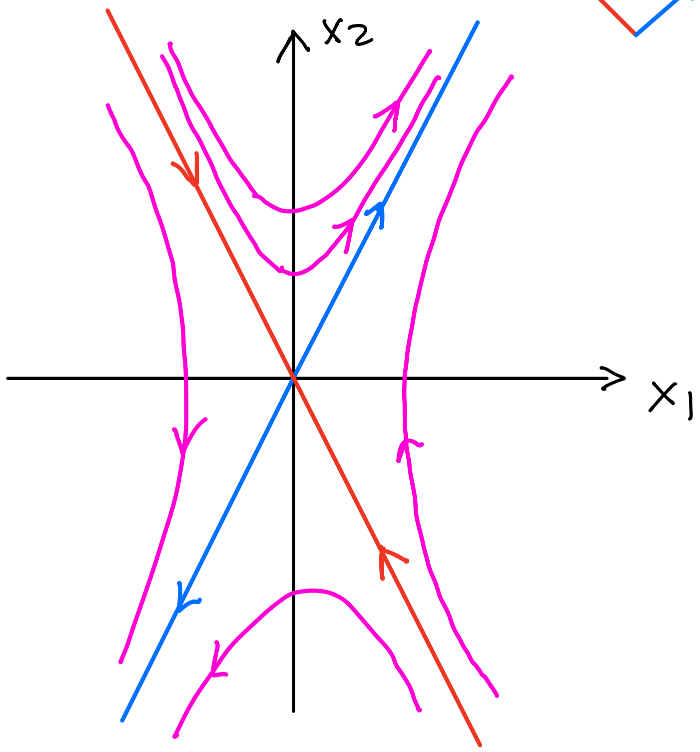
All soln's $x(t) \to \infty$ as $t \to \infty$

$x = 0$ is unstable equilibrium

Ex 3
cont'd

$$x' = \begin{bmatrix} 1 & 1 \\ 4 & 1 \end{bmatrix} x$$





	$c_1 \begin{bmatrix} 1 \\ 2 \end{bmatrix} e^{3t}$	$c_2 \begin{bmatrix} 1 \\ -2 \end{bmatrix} e^{-t}$
As $t \to \infty$	∞	0
$t \to -\infty$	0	∞

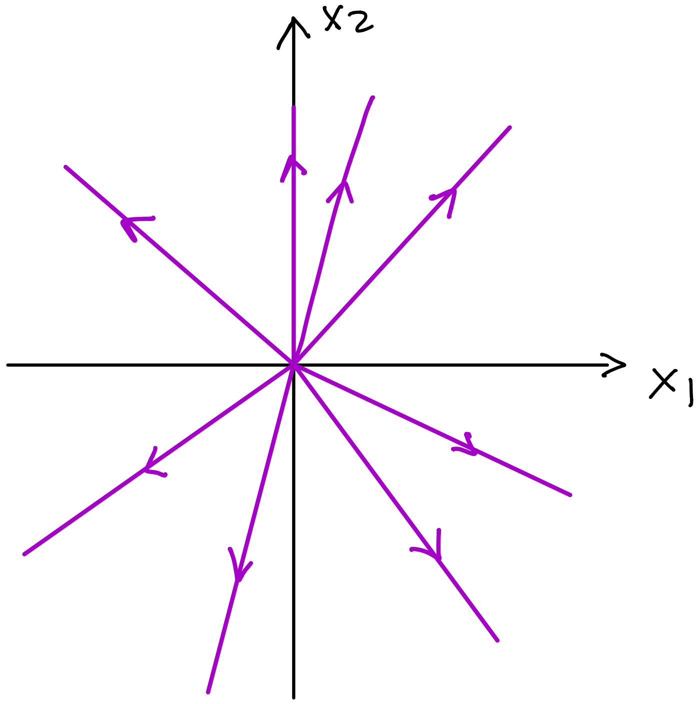
Except for $c_2 \begin{bmatrix} 1 \\ -2 \end{bmatrix} e^{-t}$,

all soln $x(t) \to \infty$ as $t \to \infty$

$x = 0$ saddle pt

Ex 4
cont'd

$$x' = \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix} x$$



$$x = \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} e^{2t} = \begin{bmatrix} c_1 e^{2t} \\ c_2 e^{2t} \end{bmatrix}$$

All directions are equally dominant.

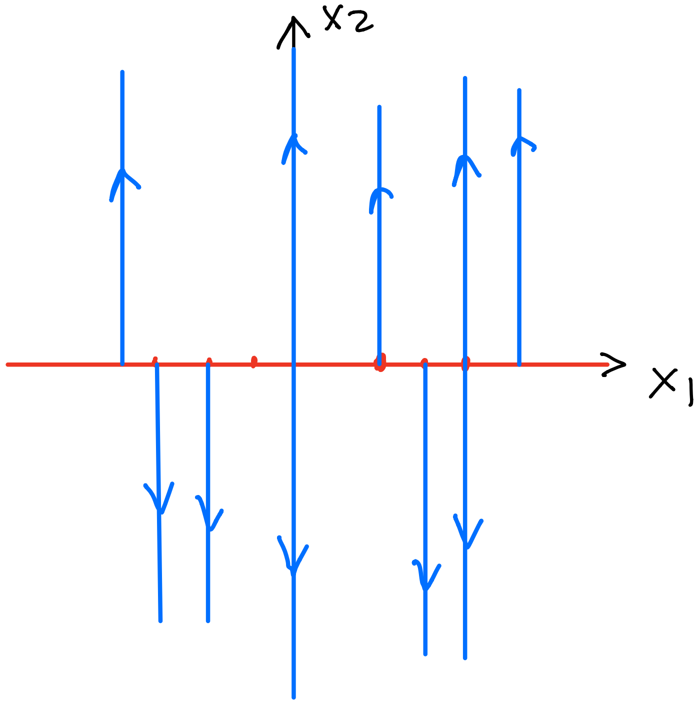
All soln's $x(t) \to \infty$ as $t \to \infty$

$x = 0$ is unstable

Equilibrium (i.e. constant) soln's to $x' = Ax$ are soln to

$$Ax = 0 \begin{cases} \text{if } \det A \neq 0, \text{ (i.e. 0 is not an eigenval)} & \text{ex 1-4} \\ \qquad x = 0 \text{ is the only constant soln} & \\ \text{if } \det A = 0, \text{ (i.e. 0 is an eigenvalue)} & \text{ex 5ab, 6} \\ \qquad \text{there are infinitely many constant soln's} & \end{cases}$$

Ex 5a



$$x' = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} x$$

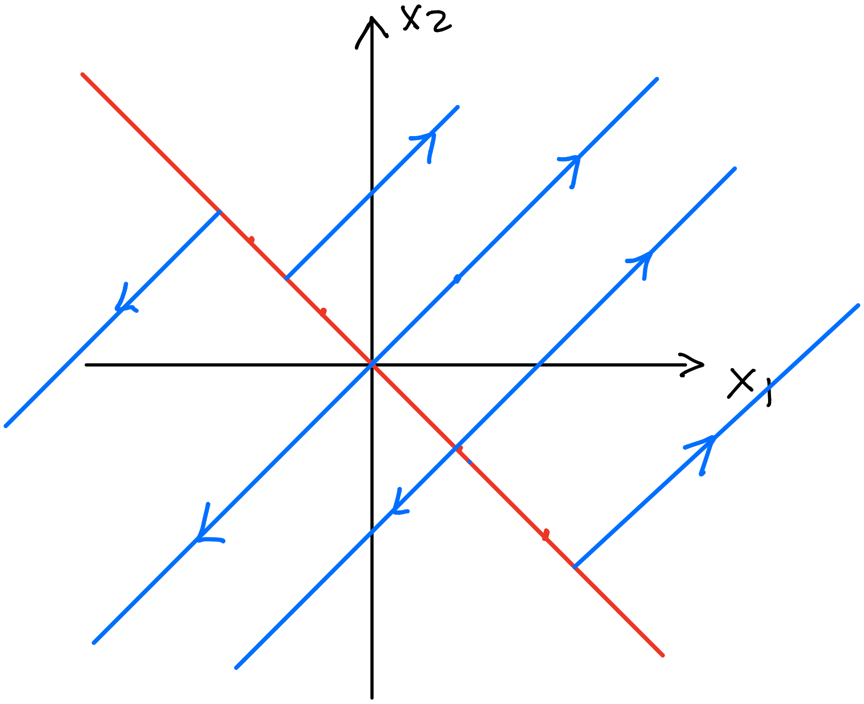
$$x = c_1 \begin{bmatrix} 1 \\ 0 \end{bmatrix} + c_2 e^{t} \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

Constant soln's : $c_1 \begin{bmatrix} 1 \\ 0 \end{bmatrix}$

each pt on the red line is itself a trajectory

All soln's are translations of $c_2 e^{t} \begin{bmatrix} 0 \\ 1 \end{bmatrix}$ by $c_1 \begin{bmatrix} 1 \\ 0 \end{bmatrix}$

Ex 5b



$$x' = \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} x$$

$$x = c_1 \begin{bmatrix} 1 \\ -1 \end{bmatrix} + c_2 e^{2t} \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

Constant soln's : $c_1 \begin{bmatrix} 1 \\ -1 \end{bmatrix}$

each pt on the red line is itself a trajectory

All soln's are translations of $c_2 e^{2t} \begin{bmatrix} 1 \\ 1 \end{bmatrix}$ by $c_1 \begin{bmatrix} 1 \\ -1 \end{bmatrix}$

What happens if

Ex6 $x' = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} x$?

Ans : x' is always 0, so all soln's are constant.

All pts $x = \begin{bmatrix} c_1 \\ c_2 \end{bmatrix}$ are constant soln's.