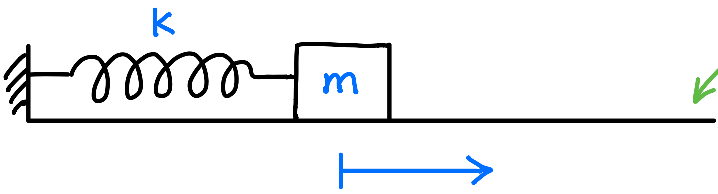
Intro. to system of differential eqn's

Harmonic Oscillator

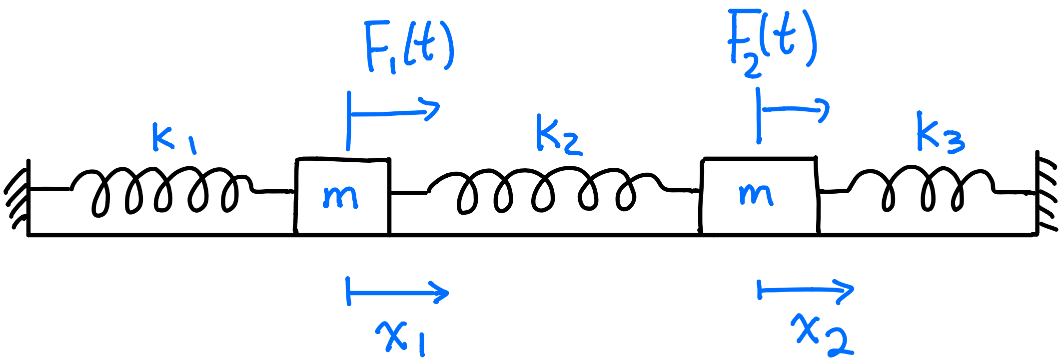
assume frictionless surface



x = displacement from equilibrium

$$m \frac{d^2x}{dt^2} = \underbrace{-kx}_{F_{spring}} \qquad \text{(Hooke's law)}$$

Coupled Oscillators



$$m \frac{d^2x_1}{dt^2} = -k_1x_1 + k_2(x_2 - x_1) + F_1(t)$$

$$= -(k_1+k_2)x_1 + k_2x_2 + F_1(t)$$

$$m \frac{d^2x_2}{dt^2} = -k_2(x_2 - x_1) - k_3x_2 + F_2(t)$$

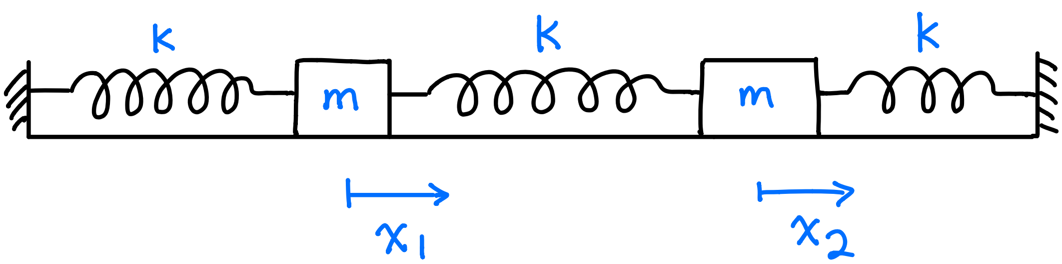
$$= k_2x_1 - (k_2+k_3)x_2 + F_2(t)$$

$$m\begin{bmatrix} x_1''(t) \\ x_2''(t) \end{bmatrix} = \underbrace{\begin{bmatrix} -(k_1+k_2) & k_2 \\ k_2 & -(k_2+k_3) \end{bmatrix}}_{A} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} F_1(t) \\ F_2(t) \end{bmatrix}$$

$$\vec{x}(t) = \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} \qquad \vec{F}(t) = \begin{bmatrix} F_1(t) \\ F_2(t) \end{bmatrix}$$

$$m \frac{d^2}{dt^2}\vec{x}(t) = A\vec{x}(t) + \vec{F}(t)$$

<u>A more specialized example</u>:



$$\begin{bmatrix} x_1'' \\ x_2'' \end{bmatrix} = -\frac{k}{m}\begin{bmatrix} 2 & -1 \\ -1 & 2 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

$$= -\frac{k}{m} P \begin{bmatrix} 1 & 0 \\ 0 & 3 \end{bmatrix} P^{-1} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

Worksheet

For $A = \begin{bmatrix} 2 & -1 \\ -1 & 2 \end{bmatrix}$

$$A = P\begin{bmatrix} 1 & 0 \\ 0 & 3 \end{bmatrix}P^{-1}$$

where

$$P = \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix}, \quad P^{-1} = \begin{bmatrix} \frac{1}{2} & \frac{1}{2} \\ -\frac{1}{2} & \frac{1}{2} \end{bmatrix}$$

$$\frac{d^2}{dt^2}\left(P^{-1}\begin{bmatrix}x_1\\x_2\end{bmatrix}\right) = P^{-1}\begin{bmatrix}x_1''\\x_2''\end{bmatrix} = \begin{bmatrix}-\frac{k}{m} & 0\\ 0 & -\frac{3k}{m}\end{bmatrix} P^{-1}\begin{bmatrix}x_1\\x_2\end{bmatrix}$$

By using eigencoordinates

$$\begin{bmatrix}y_1\\y_2\end{bmatrix} = P^{-1}\begin{bmatrix}x_1\\x_2\end{bmatrix}, \quad \text{i.e.} \quad \begin{aligned} y_1 &= \tfrac{1}{2}(x_1+x_2)\\ y_2 &= -\tfrac{1}{2}(x_1-x_2)\end{aligned}$$

We have "decoupled the system"

$$\begin{bmatrix}y_1''\\y_2''\end{bmatrix} = \begin{bmatrix}-\frac{k}{m} & 0\\ 0 & -\frac{3k}{m}\end{bmatrix}\begin{bmatrix}y_1\\y_2\end{bmatrix} \iff \begin{aligned} y_1'' &= -\tfrac{k}{m} y_1\\ y_2'' &= -\tfrac{3k}{m} y_2\end{aligned}$$

$$\left\{\begin{array}{l}
\text{From Math 307}\\
\text{char eq for } y_1'' = -\frac{k}{m} y_1 \text{ is}\\
\qquad r^2 + \frac{k}{m} = 0\\
\qquad r^2 = -\frac{k}{m}\\
\qquad r = \pm i\omega_1 \qquad\qquad \omega_1 = \sqrt{\frac{k}{m}}\\
y_1 = a_1\cos(\omega_1 t) + b_1\sin(\omega_1 t)
\end{array}\right.$$

Similary $y_2 = a_2\cos(\omega_2 t) + b_2\sin(\omega_2 t)$ $\qquad \omega_2 = \sqrt{\frac{3k}{m}}$

$$\begin{bmatrix}x_1\\x_2\end{bmatrix} = P\begin{bmatrix}y_1\\y_2\end{bmatrix}, \quad \text{i.e.} \quad \begin{aligned} x_1 &= y_1 - y_2\\ x_2 &= y_1 + y_2\end{aligned}, \quad \text{i.e.} \quad \begin{bmatrix}x_1\\x_2\end{bmatrix} = \begin{bmatrix}1\\1\end{bmatrix}y_1 + \begin{bmatrix}-1\\1\end{bmatrix}y_2$$

$$\begin{bmatrix}x_1\\x_2\end{bmatrix} = \begin{bmatrix}1\\1\end{bmatrix}\left(a_1\cos(\omega_1 t) + b_1\sin(\omega_1 t)\right) + \begin{bmatrix}-1\\1\end{bmatrix}\left(a_2\cos(\omega_2 t) + b_2\sin(\omega_2 t)\right)$$