4 (20 points) A 1 kg mass is attached to a spring with spring constant 4 Newtons/m and is forced by an external force of $5\sin(3t)$ Newtons. At time $t = 0$, the system is at the equilibrium position ($y = 0$) with initial velocity $y' = -1$ m/s.

(a) Write down the initial value problem and find the solution

$y'' + 4y = 5\sin(3t)$, $y(0)=0$, $y'(0) = -1$

char eq: $r^2+4=0 \Rightarrow r = \pm 2i$

$y(t) = C_1\cos(2t) + C_2\sin(2t) + y_p(t)$

$y_p(t) = A\sin(3t) + B\cos(3t)$

$5\sin(3t) = y_p'' + 4y_p$

$= (-9A\sin(3t) - 9B\cos 3t) + 4(A\sin(3t) + B\cos(3t))$

$= -5A\sin(3t) - 5B\cos 3t$

$\Rightarrow A=-1,\ B=0$

so,

$y(t) = C_1\cos(2t) + C_2\sin(2t) - \sin(3t)$

$0 = y(0) = C_1$

$-1 = y'(0) = (2C_2\cos(2t) - 3\cos(3t))\big|_{t=0}$

$= 2C_2 - 3$

$\Rightarrow 2C_2 = 2$

$\Rightarrow C_2 = 1$

$$\boxed{y(t) = \sin(2t) - \sin(3t)}$$

(b) Express the solution as a product and sketch the graph of the solution, illustrating any interesting phenomenon.

(Note: this is considered as a fancy trig. thing, I'll give you the formula for it if it were on the exam.

$\sin(a) - \sin(b) = 2\cos\left(\frac{a+b}{2}\right)\sin\left(\frac{a-b}{2}\right)$)

$y(t) = \sin(2t) - \sin(3t) = 2\cos\left(\frac{5}{2}t\right)\sin\left(-\frac{1}{2}t\right) = -2\cos\left(\frac{5}{2}t\right)\sin\left(\frac{1}{2}t\right)$

$\frac{1}{2} < \frac{5}{2}$, treat the wave with lower frequency as an amplitude.

so $\boxed{y(t) = -2\sin\left(\frac{1}{2}t\right)\cos\left(\frac{5}{2}t\right)}$ Phenomenon: Beats

($-2\sin\left(\frac{1}{2}t\right)$: think of this as an amplitude)

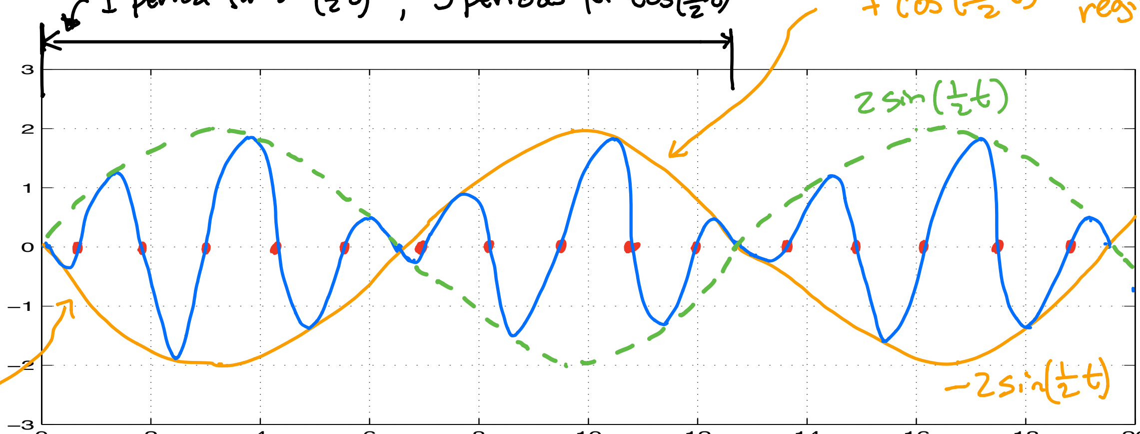
Periods

$\frac{2\pi}{1/2} = 4\pi \approx 12.4$

$\frac{2\pi}{5/2} = \frac{4\pi}{5}$

1 period for $\sin(\frac{1}{2}t)$, 5 periods for $\cos(\frac{5}{2}t)$

switching to a squeezed $+\cos(\frac{5}{2}t)$ in this region



Note: squeezed $-\cos(\frac{5}{2}t)$ in this region: follow the sign of $-2\sin(\frac{1}{2}t)$

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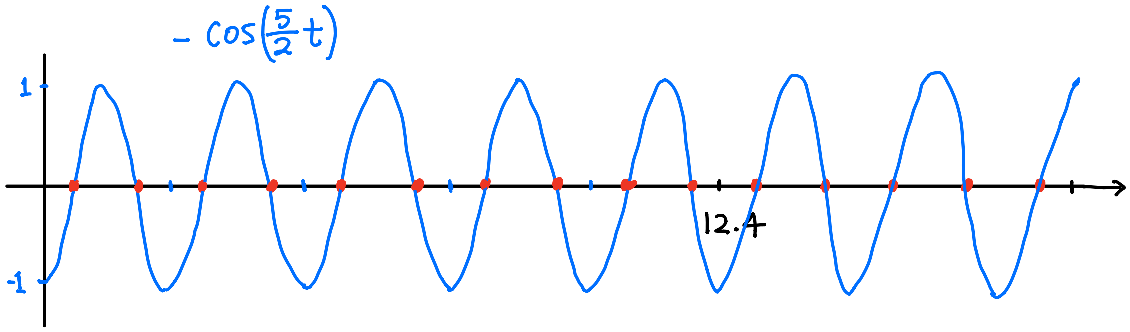

When squeezing $\pm\cos\left(\frac{5}{2}t\right)$ into the amplitude envelope, it's easiest if you try to match where the zeros of the $\pm\cos\left(\frac{5}{2}t\right)$ goes as shown by the red dots.