Gallery: all the graphs have appeared in lectures, details see lectures

① Harmonic Oscillator

$$\boxed{mu'' + ku = 0}$$

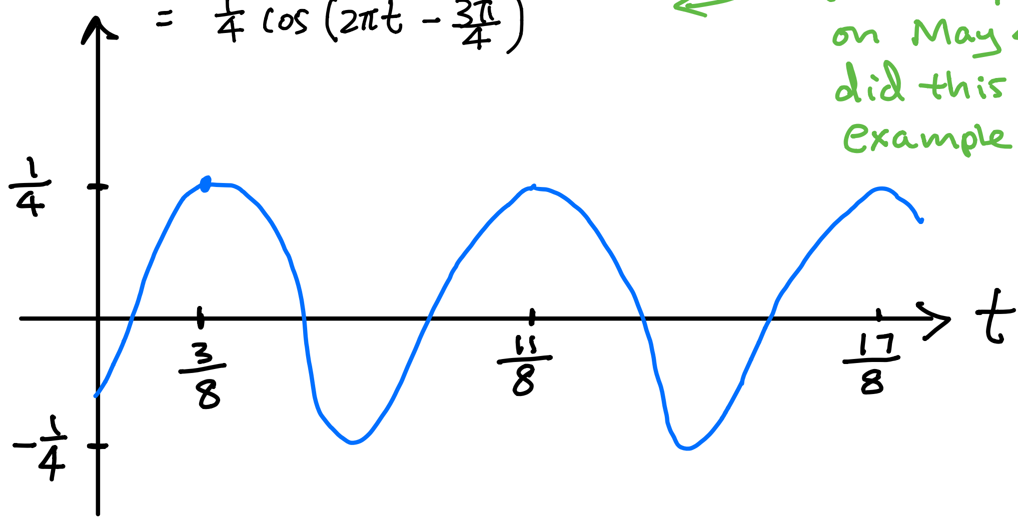
<u>Example</u>:

$$u'' + 4\pi^2 u = 0, \quad u(0) = -\frac{1}{4\sqrt{2}}, \quad u'(0) = \frac{1}{4\sqrt{2}}$$

$$u(t) = -\frac{1}{4\sqrt{2}}\cos(2\pi t) + \frac{1}{4\sqrt{2}}\sin(2\pi t)$$

$$= \frac{1}{4}\cos\left(2\pi t - \frac{3\pi}{4}\right)$$

If you are not sure how to do this step, see lecture on May 4, where I did this particular example.



* First peak when $2\pi t - \frac{3\pi}{4} = 0$

$$2\pi t = \frac{3\pi}{4}$$

$$t = \frac{3}{8}$$

* $u(0) < 0$

* Period: $T = \frac{2\pi}{\omega_0} = \frac{2\pi}{2\pi} = \underline{1}$

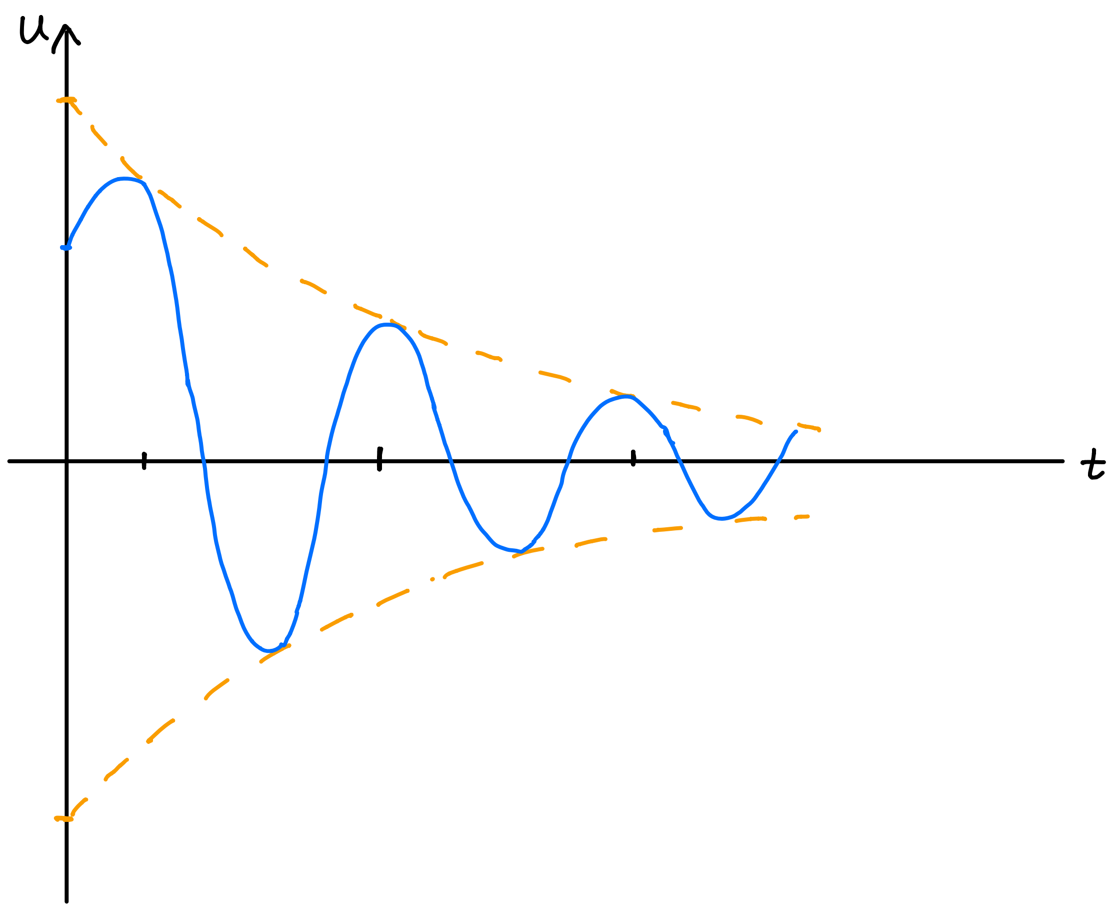
② Damped Harmonic Oscillators

$$\boxed{mu'' + \gamma u' + ku = 0}$$

Note: damped vibrations without driving force always decays exponentially

Char eqn : $mr^2 + \gamma r + k = 0$

Underdamped case : when char eqn has complex roots



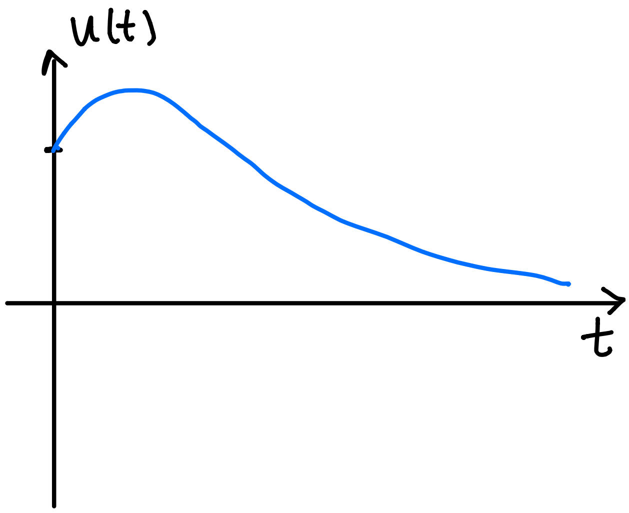
Key features: * Amplitude envelope

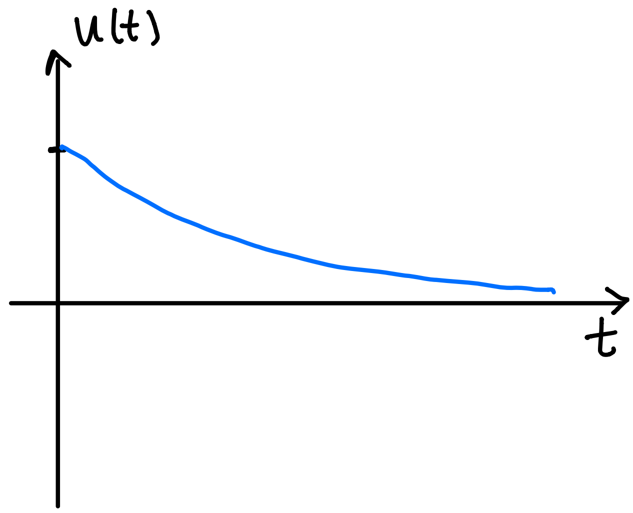
* sign of $u(0)$
* First time it touches the upper envelope
* period

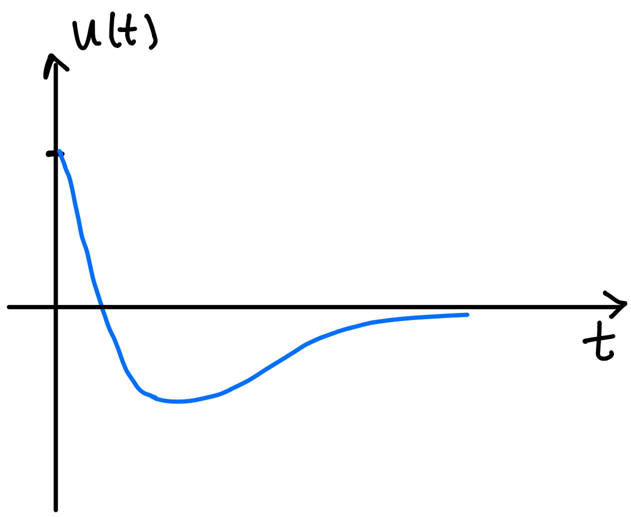
Critically damped case: when the char eqn has one real root (i.e. repeated real roots)

Overdamped case when the char eqn has two distinct real roots

Critical damping and overdamping have similar looking graphs, here are some samples of what the soln curve could look like:







③ Undamped Forced Vibration

$$\boxed{m u'' + k u = F_0 \cos(\omega t)}$$ (or similarly $F_0 \sin(\omega t)$)

Char eqn : $m r^2 + k = 0$

$$r^2 = -\frac{k}{m}$$

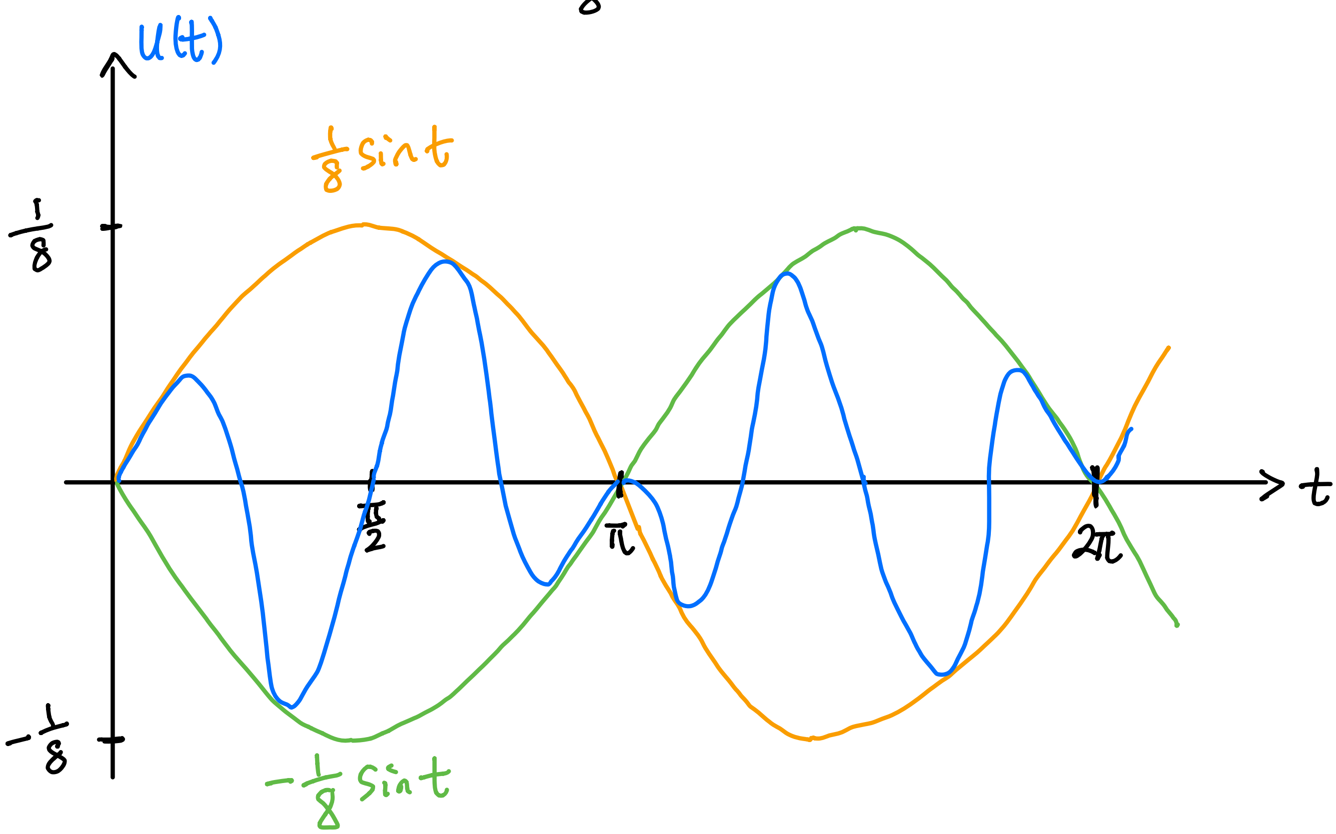
$$r = \pm i \omega_0 , \quad \omega_0 = \sqrt{\frac{k}{m}}$$

When $\omega_0 \neq \omega \Rightarrow$ Beats

Example $u'' + 25u = \cos 3t$, $u(0) = 0$, $u'(0) = 0$

Soln: $u(t) = -\frac{1}{16}\cos 5t + \frac{1}{16}\cos 3t$

$$= \frac{1}{8} \sin t \sin 4t$$



When $\omega_0 = \omega \Rightarrow$ Resonance

Example $u'' + u = \cos t$, $u(0) = 2$, $u'(0) = 1$

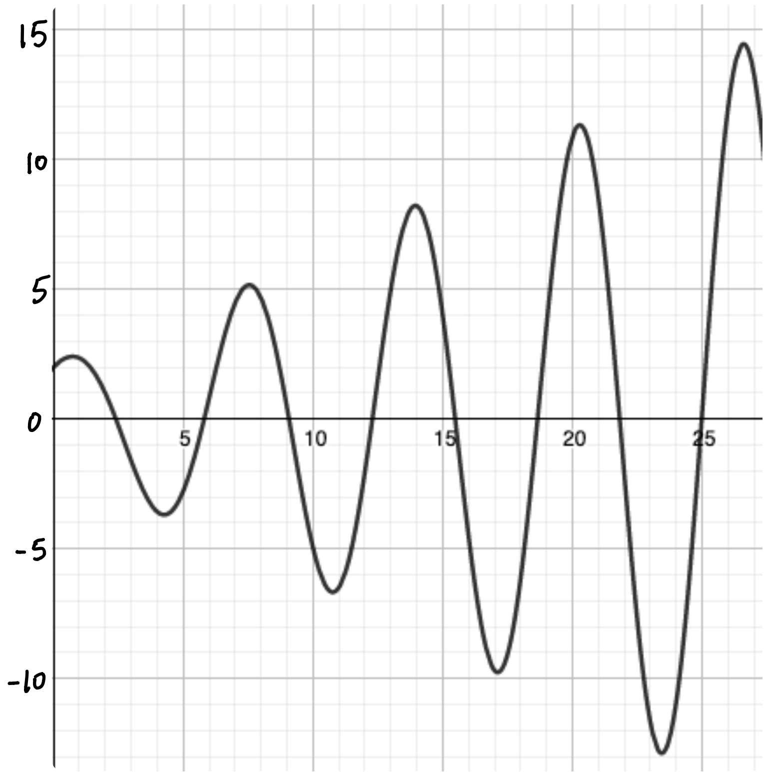
$$u(t) = \underbrace{2\cos t + \sin t}_{\text{soln to the homog. part}} + \underbrace{\tfrac{1}{2} t \sin t}_{\text{particular soln.}}$$

Graph of the full soln $u(t)$:

This graph is eventually dominated by the particular soln $\frac{1}{2} t \sin t$ whose amplitude $\to \infty$ as $t \to \infty$

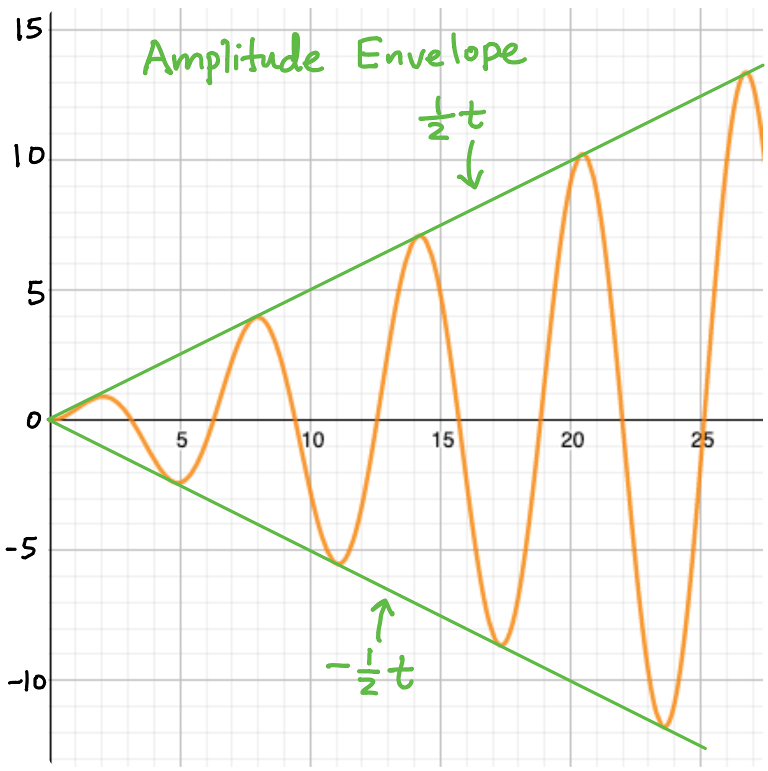
You won't be required to draw this full soln by hand

But you should be able to recognize that this kind of graph corresponds to resonance without damping



Graph of the particular soln $\frac{1}{2} t \sin t$:

You should be able to sketch this by hand. Just draw $\sin t$ and fit that in the envelope $\pm \frac{1}{2} t$

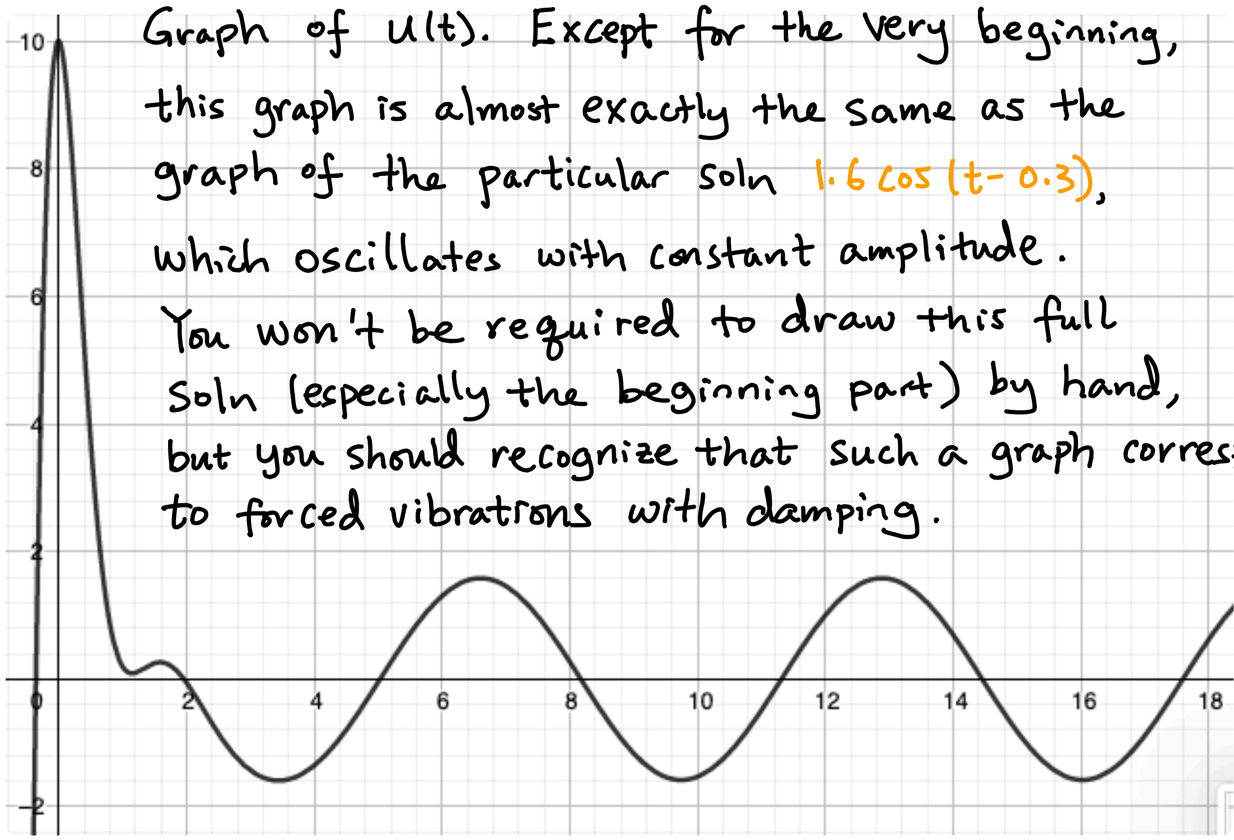


④ Damped Forced Vibrations

$$u'' + 4u' + 13u = 20\cos(t), \quad u(0) = 10, \quad u'(0) = 0$$

$$u(t) = \overbrace{8.5e^{-2t}\cos(3t) + 5.5e^{-2t}\sin(3t)}^{\text{soln to the homog. part}} + \underbrace{\overbrace{\tfrac{3}{2}\cos t + \tfrac{1}{2}\sin t}^{\text{Particular soln}}}_{=1.6\cos(t-0.3)}$$

Graph of $u(t)$. Except for the very beginning, this graph is almost exactly the same as the graph of the particular soln $1.6\cos(t-0.3)$, which oscillates with constant amplitude.
You won't be required to draw this full soln (especially the beginning part) by hand, but you should recognize that such a graph corresponds to forced vibrations with damping.



Below is the graph of the particular soln $1.6\cos(t-0.3)$. You should be able to draw this particular soln part by hand (same kind of graph as harmonic oscillators, but at at the driving frequency)