§6.6

Property 6 $\quad F(s)=\mathcal{L}\{f(t)\}, \quad G(s)=\mathcal{L}\{g(t)\}$

$$\boxed{\mathcal{L}^{-1}\{F(s)G(s)\} = (f*g)(t)}$$

(Convolution)

$$= \int_0^t f(t-\tau)g(\tau)\,d\tau$$

$$= \int_0^t f(\tau)g(t-\tau)\,d\tau$$

$$\int_0^t f(t-\tau)g(\tau)\,d\tau$$

$u = t-\tau$

$du = -d\tau$

$$= -\int_t^0 f(u)g(t-u)\,du$$

$$= \int_0^t f(u)g(t-u)\,du$$

(A proof of Property 6 is provided on the last page)

Ex1 $\quad f(t)=\cos t, \quad g(t)=1$

$$(f*g)(t) = \int_0^t f(\tau)g(t-\tau)\,d\tau = \int_0^t \cos\tau\,d\tau = \sin\tau\Big|_0^t = \sin t$$

or

$$(f*g)(t) = \int_0^t f(t-\tau)g(\tau)\,d\tau = \int_0^t \cos(t-\tau)\,d\tau = -\sin(t-\tau)\Big|_0^t = \sin t$$

Ex2: $\quad F(s) = \dfrac{1}{2s^2+s+2}, \quad f(t) = \dfrac{2}{\sqrt{15}}e^{-\frac{t}{4}}\sin\left(\dfrac{\sqrt{15}}{4}t\right)$

(From the last lecture)

$$G(s) = e^{-5s}, \quad g(t) = \delta(t-5)$$

$$\mathcal{L}^{-1}\{F(s)G(s)\} = (f*g)(t)$$

$$= \int_0^t f(t-\tau)g(\tau)\,d\tau$$

$$= \int_0^t \frac{2}{\sqrt{15}} e^{-\frac{(t-\tau)}{4}} \sin\left(\frac{\sqrt{15}}{4}(t-\tau)\right)\delta(\tau-5)\,d\tau$$

$$= \begin{cases} \frac{2}{\sqrt{15}} e^{-\frac{(t-5)}{4}} \sin\left(\frac{\sqrt{15}}{4}(t-5)\right), & t \geq 5 \\ 0, & t < 5 \end{cases}$$

$$= u_5(t)\,\frac{2}{\sqrt{15}} e^{-\frac{(t-5)}{4}} \sin\left(\frac{\sqrt{15}}{4}(t-5)\right)$$

Immediate Properties

$$f*g = g*f$$

$$f*(g_1+g_2) = f*g_1 + f*g_2$$

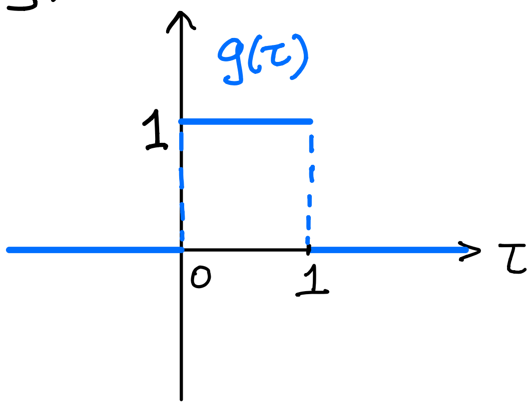
$$(f*g)*h = f*(g*h)$$

$$0*f = f*0 = 0$$

Warning: $f*1 \neq f$, e.g. $(\cos t)*1 = \sin t$

More about $f*g$ (Geometric meaning)

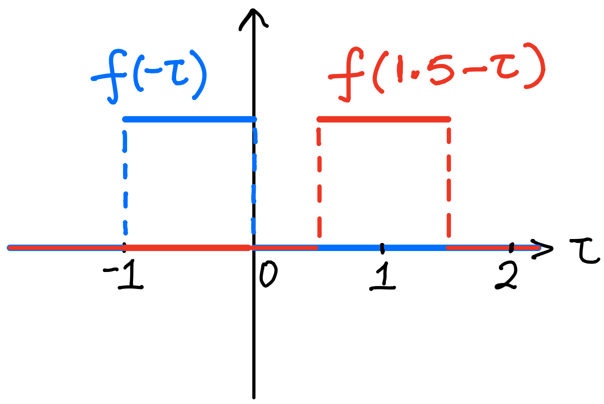
Ex: $f(t) = g(t) = u_0(t) - u_1(t)$

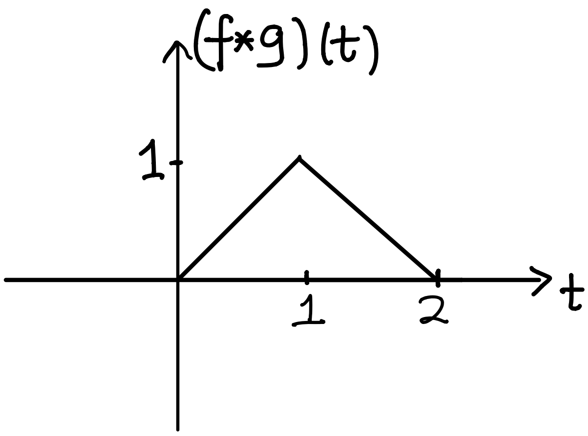


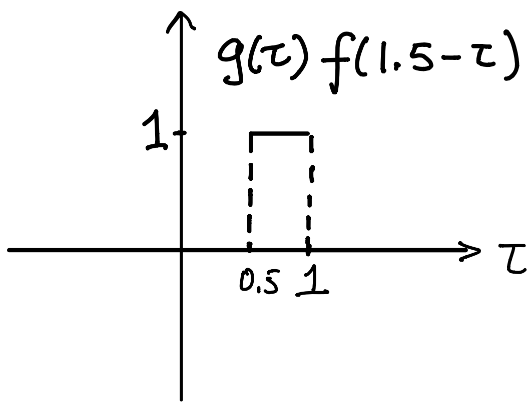
$$(f*g)(t) = \int_0^t f(t-\tau)\, g(\tau)\, d\tau$$

$$= \text{Area of overlap}$$

$$= \begin{cases} 0, & t<0 \\ t, & 0 \le t \le 1 \\ 2-t, & 1 \le t \le 2 \\ 0, & t>2 \end{cases}$$







Ex: Solve $2y'' + y' + 2y = \delta(t-5)$, $y(0)=0$, $y'(0)=0$

$$\mathcal{L}\{2y'' + y' + 2y\} = \mathcal{L}\{\delta(t-5)\}$$

$\vdots$ (see last lecture)

$$\underbrace{(2s^2+s+2)}_{\text{char eq}}Y(s) = e^{-5s}$$

$$Y(s) = F(s)\,G(s)$$

$$F(s) = \frac{1}{2s^2+s+2}$$ Transfer function (only depends on the spring-mass system)

$$G(s) = e^{-5s}$$ Only depends on the external force

$$y(t) = \mathcal{L}^{-1}\{F(s)G(s)\}$$ Green's function for $2y'' + y' + 2y$

$$= (f * g)(t)$$

$$= u_5(t)\,\frac{2}{\sqrt{15}}\,e^{-\frac{(t-5)}{4}}\sin\left(\frac{\sqrt{15}}{4}(t-5)\right)$$

(More on next page)

Proof of Property 6

$$F(s)\,G(s) = \left(\int_0^\infty e^{-su} f(u)\,du\right)\left(\int_0^\infty e^{-s\tau} g(\tau)\,d\tau\right)$$

$$= \int_0^\infty e^{-s\tau} g(\tau) \left(\int_0^\infty e^{-su} f(u)\,du\right) d\tau$$

$$= \int_0^\infty g(\tau) \left(\int_0^\infty e^{-s(u+\tau)} f(u)\,du\right) d\tau$$

$$= \int_0^\infty g(\tau) \left(\int_\tau^\infty e^{-st} f(t-\tau)\,dt\right) d\tau$$

$t = u+\tau$

$du = dt$

$$= \int_0^\infty e^{-st} \underbrace{\left(\int_0^t f(t-\tau)\, g(\tau)\,d\tau\right)}_{(f*g)(t)} dt$$

$$= \int_0^\infty e^{-st} (f*g)(t)\,dt$$

$$= \mathcal{L}\{(f*g)(t)\}$$