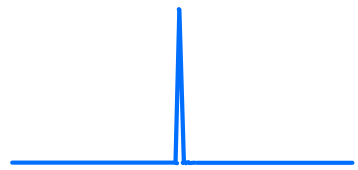
Impulse Function / Dirac Delta Function §6.5

$$my'' + \gamma y' + ky = \text{impulse function}$$



$t = 5$

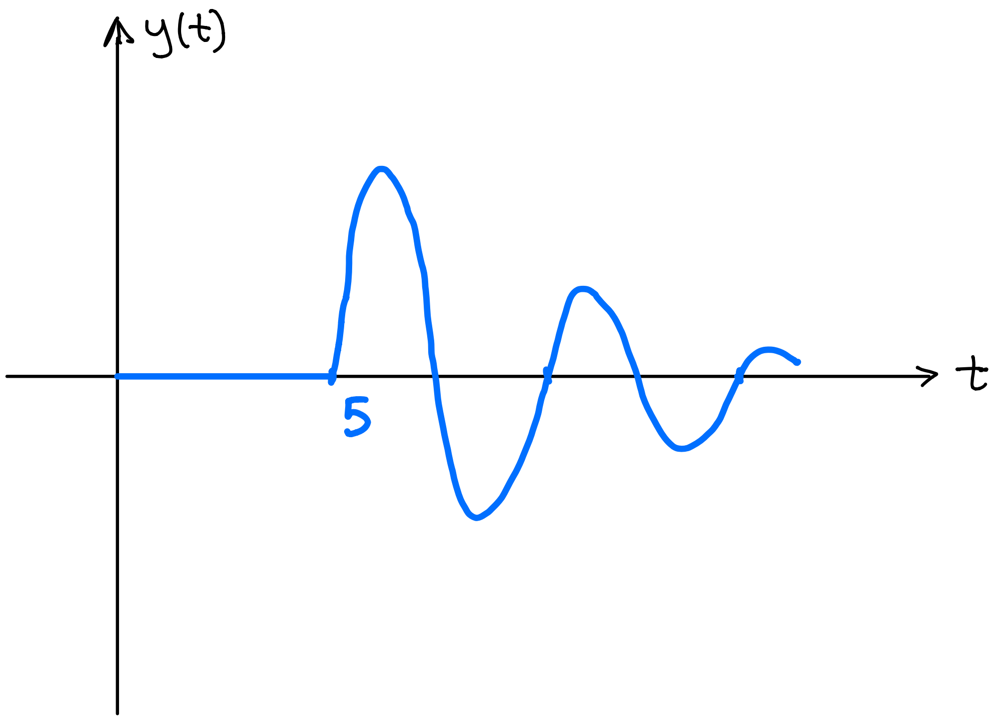
Ex: $2y'' + y' + 2y = \delta(t-5)$, $y(0) = 0$, $y'(0) = 0$

$2y'' + y' + 2y = 0$, for $t \neq 5$ $\quad y(5) = 0$, $y'(5) \neq 0$

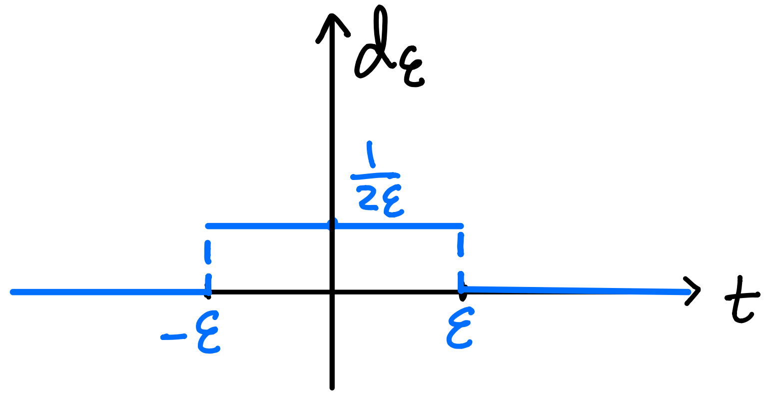
<u>char eq</u>: $2r^2 + r + 2 = 0$

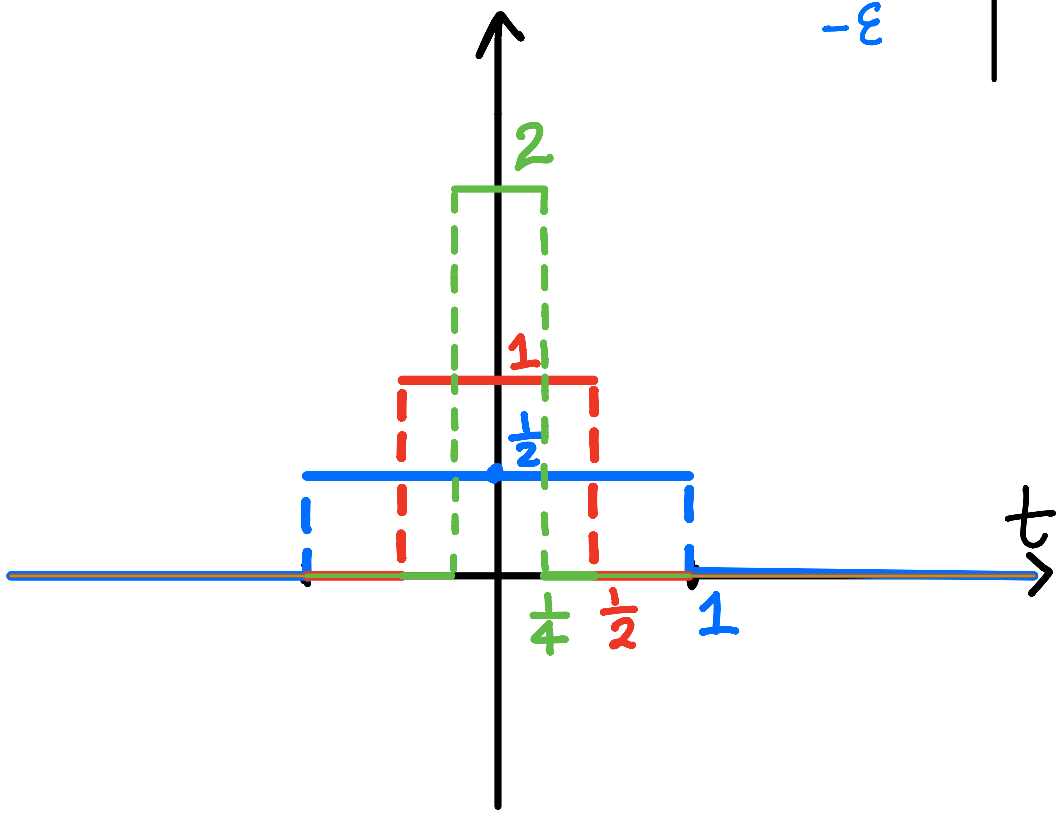
$$r = \frac{-1 \pm \sqrt{1-16}}{4} = -\frac{1}{4} \pm \frac{\sqrt{-15}}{4} = -\frac{1}{4} \pm i\frac{\sqrt{15}}{4}$$

$$y(t) = u_5(t)\, A e^{-\frac{1}{4}(t-5)} \sin\left(\frac{\sqrt{15}}{4}(t-5)\right)$$



$$d_\varepsilon(t) = \begin{cases} \frac{1}{2\varepsilon}, & -\varepsilon < t < \varepsilon, \quad \varepsilon > 0 \\ 0, & \text{otherwise} \end{cases}$$





$$\int_{-\infty}^{\infty} d_\varepsilon(t)\,dt = 2\varepsilon\left(\frac{1}{2\varepsilon}\right) = 1$$

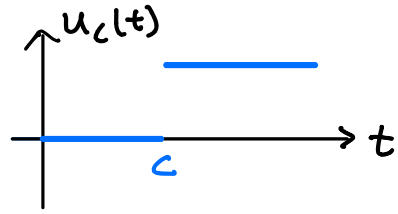
As $\varepsilon \to 0$, $\quad \frac{1}{2\varepsilon} \to \infty$

$d_\varepsilon(t) \to \delta(t)$ = Dirac Delta Function
(Impulse function)

$\underline{\text{Def}^n}$: $\left.\begin{cases} \delta(t) = 0 \quad \text{for} \quad t \neq 0 \\ \int_{-\infty}^{\infty} \delta(t)\,dt = 1 \end{cases}\right\}$ generalized function/distribution

Note: $\delta(t-c)$ is centered at to.

Fact 1: $\frac{d}{dt} u_c(t) = \delta(t-c)$



Fact 2:

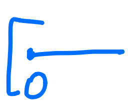
$$\int_{-\infty}^{\infty} \delta(t-c) f(t)\, dt \qquad \text{(Assume } f(t) \text{ is continuous)}$$

$$= \int_{-\infty}^{\infty} f(c)\, \delta(t-c)\, dt = f(c) \int_{-\infty}^{\infty} \delta(t-c)\, dt = f(c)$$

$$\boxed{\int_{-\infty}^{\infty} \delta(t-c) f(t)\, dt = f(c)}$$

$$\mathcal{L}\{\delta(t-c)\} = \int_0^{\infty} \delta(t-c)\, e^{-st}\, dt$$

$$= \begin{cases} (\text{if } c < 0) & 0 \\ (\text{if } c > 0) & \int_{-\infty}^{\infty} \delta(t-c)\, e^{-st}\, dt = e^{-sc} \\ (\text{if } c = 0) & \text{Define } \mathcal{L}\{\delta(t)\} = \int_{0^-}^{\infty} \delta(t)\, e^{-st}\, dt \end{cases}$$



Then $\mathcal{L}\{\delta(t)\} = \int_{-\infty}^{\infty} \delta(t)\, e^{-st}\, dt$

$$= e^{-s0} = e^{0} = 1$$

$$\boxed{\mathcal{L}\{\delta(t-c)\} = \begin{cases} e^{-sc}, & \text{if } c \geq 0 \\ 0, & \text{if } c < 0 \end{cases}}$$

Ex: $2y'' + y' + 2y = \delta(t-5)$, $y(0)=0$, $y'(0)=0$

$$2\left(s^2 Y(s) - s\cancel{y(0)} - \cancel{y'(0)}\right) + \left(sY(s) - \cancel{y(0)}\right) + 2Y(s) = \mathcal{L}\{\delta(t-5)\}$$

$$(2s^2+s+2)Y(s) = e^{-5s}$$

$$Y(s) = \frac{e^{-5s}}{2s^2+s+2} = e^{-5s}H(s)$$

$$H(s) = \frac{1}{2s^2+s+2}$$

$$h(t) = \mathcal{L}^{-1}\{H(s)\}$$

$$\boxed{y(t) = \mathcal{L}^{-1}\{Y(s)\} = \mathcal{L}^{-1}\{e^{-5s}H(s)\} = u_5(t)h(t-5)}$$

$$H(s) = \frac{1}{2s^2+s+2} = \frac{1}{2}\,\frac{1}{s^2+\frac{s}{2}+1}$$

$$= \frac{1}{2}\,\frac{1}{\left(s+\frac{1}{4}\right)^2 + \left(\frac{\sqrt{15}}{4}\right)^2}$$

$$= \frac{1}{2}\,\frac{1}{\frac{\sqrt{15}}{4}}\,\frac{\frac{\sqrt{15}}{4}}{(s+\frac{1}{4})^2+\left(\frac{\sqrt{15}}{4}\right)^2}$$

$$= \frac{2}{\sqrt{15}}\,\frac{\frac{\sqrt{15}}{4}}{(s+\frac{1}{4})^2+\left(\frac{\sqrt{15}}{4}\right)^2}$$

$$= \frac{2}{\sqrt{15}}\,\mathcal{L}\left\{e^{-\frac{t}{4}}\sin\left(\frac{\sqrt{15}}{4}t\right)\right\}$$

$$h(t) = \mathcal{L}^{-1}\{H(s)\} = \frac{2}{\sqrt{15}}\,e^{-\frac{t}{4}}\sin\left(\frac{\sqrt{15}}{4}t\right)$$

$$\boxed{\begin{aligned} y(t) &= \mathcal{L}^{-1}\{Y(s)\} = u_5(t)\,h(t-5) \\ &= u_5(t)\frac{2}{\sqrt{15}}\,e^{-\frac{t-5}{4}}\sin\left(\frac{\sqrt{15}}{4}(t-5)\right) \end{aligned}}$$

Exercise added after the lecture:

Solve $2y'' + y' + 2y = \delta(t-5) + \delta(t-10)$, $y(0)=0$, $y'(0)=0$

Answer: $Y(s) = (e^{-5s} + e^{-10s})H(s)$

so $y(t) = u_5(t)\,h(t-5) + u_{10}(t)\,h(t-10)$

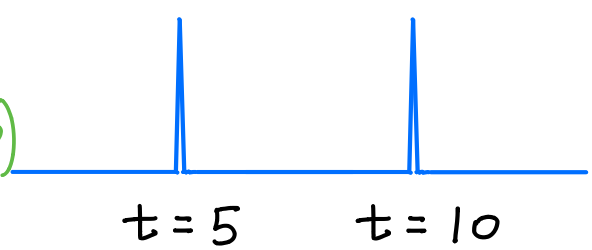


Table of Laplace Transforms

	$f(t) = \mathcal{L}^{-1}\{F(s)\}$	$F(s) = \mathcal{L}\{f(t)\}$
1.	1	$\frac{1}{s}\,,\quad s > 0$
2.	e^{at}	$\frac{1}{s-a},\quad s > a$
3.	$\sinh at = \frac{e^{at}-e^{-at}}{2}$	$\frac{a}{s^2-a^2}\,,\quad s > \lvert a\rvert$
4.	$\cosh at = \frac{e^{at}+e^{-at}}{2}$	$\frac{s}{s^2-a^2}\,,\quad s > \lvert a\rvert$
5.	$t^n,\quad n =$positive integer	$\frac{n!}{s^{n+1}},\ s > 0$
6.	$t^n e^{at},\quad n =$ positive integer	$\frac{n!}{(s-a)^{n+1}},\quad s > a$
7.	$\sin bt$	$\frac{b}{s^2+b^2},\quad s > 0$
8.	$\cos bt$	$\frac{s}{s^2+b^2},\quad s > 0$
9.	$e^{at}\sin bt$	$\frac{b}{(s-a)^2+b^2},\quad s > a$
10.	$e^{at}\cos bt$	$\frac{s-a}{(s-a)^2+b^2},\quad s > a$
11.	$u_c(t)$	$\frac{e^{-cs}}{s},\ s > 0$
12.	$u_c(t)f(t-c)$	$e^{-cs}F(s)$
13.	$e^{ct}f(t)$	$F(s-c)$
14.	$\delta(t-c)$	e^{-cs} when $c \geq 0$; 0 when $c < 0$
15.	$f^{(n)}(t)$	$s^nF(s) - s^{n-1}f(0) - \cdots - f^{(n-1)}(0)$
16.	$(-t)^n f(t)$	$F^{(n)}(s)$
17.	$\int_0^t f(t-\tau)g(\tau)d\tau$	$F(s)G(s)$

$$\frac{1}{2s^2+s+2}$$

$$= \frac{1}{2}\,\frac{1}{s^2+\frac{s}{2}+1}$$

$$= \frac{1}{2}\,\frac{1}{\left(s+\frac{1}{4}\right)^2+\left(\frac{\sqrt{15}}{4}\right)^2}$$

$$= \frac{1}{2}\,\frac{1}{\frac{\sqrt{15}}{4}}\,\frac{\frac{\sqrt{15}}{4}}{\left(s+\frac{1}{4}\right)^2+\left(\frac{\sqrt{15}}{4}\right)^2}$$

$$= \frac{2}{\sqrt{15}}\,\frac{\frac{\sqrt{15}}{4}}{\left(s+\frac{1}{4}\right)^2+\left(\frac{\sqrt{15}}{4}\right)^2}$$

$$= \frac{2}{\sqrt{15}}\,\mathcal{L}\left\{e^{-\frac{t}{4}}\sin\left(\frac{\sqrt{15}}{4}t\right)\right\}$$