Laplace Transform

$$\boxed{\int_0^\infty f(t)e^{-st}\,dt = F(s) = \mathcal{L}\{f(t)\} \quad \text{for all } s \text{ s.t. the integral is defined}}$$

Property 1 (Linearity)

$$\boxed{\mathcal{L}\{c_1 f_1(t) + c_2 f_2(t)\}}$$

$$= \int_0^\infty [c_1 f_1(t) + c_2 f_2(t)]e^{-st}\,dt$$

$$= c_1 \int_0^\infty f_1(t)e^{-st}\,dt + c_2 \int_0^\infty f_2(t)e^{-st}\,dt$$

$$\boxed{= c_1 \mathcal{L}\{f_1(t)\} + c_2 \mathcal{L}\{f_2(t)\}}$$

Property 2

$$\mathcal{L}\{f'(t)\} = \int_0^\infty f'(t)e^{-st}\,dt$$

$$\overset{\text{i.b.p.}}{=} f(t)e^{-st}\Big|_0^\infty + s\int_0^\infty f(t)e^{-st}\,dt$$

$$= (\underset{\substack{\uparrow \\ \text{if limit} \\ \text{exists}}}{0} - f(0)) + s\mathcal{L}\{f(t)\}$$

$$\boxed{\mathcal{L}\{f'(t)\} = s\mathcal{L}\{f(t)\} - f(0)}$$

Ex: last time $\mathcal{L}\{\sin(bt)\} = \frac{b}{s^2+b^2}$

$$\frac{d}{dt}(\sin t) = \cos t$$

$$\begin{aligned}
\mathcal{L}\{\cos t\} &= \mathcal{L}\left\{\frac{d}{dt}(\sin t)\right\} \\
&= s\,\mathcal{L}\{\sin t\} - \left(\sin t\big|_{t=0}\right) \\
&= s \cdot \frac{1}{s^2+1} - 0 \\
&= \frac{s}{s^2+1}
\end{aligned}$$

Property 3

$$\begin{aligned}
\boxed{\mathcal{L}\{t f(t)\}} &= \int_0^\infty t f(t)\, e^{-st}\, dt \\
&= \int_0^\infty \frac{d}{ds}\left(-f(t) e^{-st}\right) dt \\
&= -\frac{d}{ds}\int_0^\infty f(t)\, e^{-st}\, dt \\
&\boxed{= -\frac{d}{ds}\left(\mathcal{L}\{f(t)\}\right)}
\end{aligned}$$

Ex: $\mathcal{L}\{t\} = \mathcal{L}\{t \cdot 1\}$

$$= -\frac{d}{ds}\left(\mathcal{L}\{1\}\right) = -\frac{d}{ds}\left(\frac{1}{s}\right) = \frac{1}{s^2}$$

Property 2 Cont'd

$$\mathcal{L}\{f'(t)\} = s\mathcal{L}\{f(t)\} - f(0)$$

$$\mathcal{L}\{f''(t)\} = s\mathcal{L}\{f'(t)\} - f'(0)$$

$$= s\left(s\mathcal{L}\{f(t)\} - f(0)\right) - f'(0)$$

$$= s^2\mathcal{L}\{f(t)\} - sf(0) - f'(0)$$

$\vdots$

$$\mathcal{L}\{f^{(n)}(t)\} = s^n\mathcal{L}\{f(t)\} - s^{n-1}f(0) - s^{n-2}f'(0) - \cdots - sf^{(n-2)}(0) - f^{(n-1)}(0)$$

<u>Ex</u>: $y'' + y = \sin(2t)$, $y(0) = 2$, $y'(0) = 1$

Denote $Y(s) = \mathcal{L}\{y(t)\}$

$$\mathcal{L}\{y'' + y\} = \mathcal{L}\{\sin(2t)\}$$

$$\mathcal{L}\{y''\} + \mathcal{L}\{y\} = \mathcal{L}\{\sin(2t)\}$$

$$\left(s^2 Y(s) - sy(0) - y'(0)\right) + Y(s) = \frac{2}{s^2+4}$$

$$s^2 Y(s) - 2s - 1 + Y(s) = \frac{2}{s^2+4}$$

$$\underbrace{(s^2+1)}_{\text{same as characteristic poly}} Y(s) = \frac{2}{s^2+4} + 2s + 1$$

$Y(s)$ satisfies an algebraic eqn not a diff. eqn

$$Y(s) = \frac{2s^3+s^2+8s+6}{(s^2+4)(s^2+1)}$$

$y(t) = \mathcal{L}^{-1}\{Y(s)\}$ ← unique: if f, g continous functions, $\mathcal{L}\{f\} = \mathcal{L}\{g\}$, the $f = g$

partial fration:

$$Y(s) = \frac{as+b}{s^2+1} + \frac{cs+d}{s^2+4}$$

$$= \frac{(as+b)(s^2+4) + (cs+d)(s^2+1)}{(s^2+1)(s^2+4)}$$

$$= \frac{(a+c)s^3 + (b+d)s^2 + (4a+c)s + (4b+d)}{(s^2+1)(s^2+4)}$$

$\Rightarrow$ $a+c=2$ $b+d=1$
$4a+c=8$ $4b+d=6$

4 eqn 4 unknows

$\Rightarrow$ $a=2$, $c=0$, $b=\frac{5}{3}$, $d=-\frac{2}{3}$

$$Y(s) = \frac{2s}{s^2+1} + \frac{5/3}{s^2+1} - \frac{2/3}{s^2+4}$$

$$y(t) = \mathcal{L}^{-1}\{Y(s)\} = 2\cos t + \frac{5}{3}\sin t - \frac{1}{3}\sin(2t)$$

Table of Laplace Transforms

	$f(t) = \mathcal{L}^{-1}\{F(s)\}$	$F(s) = \mathcal{L}\{f(t)\}$
1.	1	$\frac{1}{s}\ ,\quad s > 0$
2.	e^{at}	$\frac{1}{s-a},\quad s > a$
3.	$\sinh at = \frac{e^{at}-e^{-at}}{2}$	$\frac{a}{s^2-a^2}\ ,\quad s > \lvert a \rvert$
4.	$\cosh at = \frac{e^{at}+e^{-at}}{2}$	$\frac{s}{s^2-a^2}\ ,\quad s > \lvert a \rvert$
5.	$t^n, \quad n =$positive integer	$\frac{n!}{s^{n+1}},\ s > 0$
6.	$t^n e^{at}, \quad n =$ positive integer	$\frac{n!}{(s-a)^{n+1}},\quad s > a$
7.	$\sin bt$	$\frac{b}{s^2+b^2},\quad s > 0$
8.	$\cos bt$	$\frac{s}{s^2+b^2},\quad s > 0$
9.	$e^{at}\sin bt$	$\frac{b}{(s-a)^2+b^2},\quad s > a$
10.	$e^{at}\cos bt$	$\frac{s-a}{(s-a)^2+b^2},\quad s > a$
11.	$u_c(t)$	$\frac{e^{-cs}}{s},\ s > 0$
12.	$u_c(t)f(t-c)$	$e^{-cs}F(s)$
13.	$e^{ct}f(t)$	$F(s-c)$
14.	$\delta(t-c)$	e^{-cs} when $c \geq 0$; 0 when $c < 0$
15.	$f^{(n)}(t)$	$s^n F(s) - s^{n-1}f(0) - \cdots - f^{(n-1)}(0)$
16.	$(-t)^n f(t)$	$F^{(n)}(s)$
17.	$\int_0^t f(t-\tau)g(\tau)d\tau$	$F(s)G(s)$

$$\frac{2/3}{s^2+4}$$

$$\frac{2}{3}\cdot\frac{1}{s^2+4}$$

$$\frac{2}{3}\cdot\frac{1}{2}\,\frac{2}{s^2+4}$$

$$= \left(\frac{2}{3}\cdot\frac{1}{2}\right)\frac{2}{s^2+4}$$

$$= \frac{1}{3}\,\frac{2}{s^2+4}$$

$$= \frac{1}{3}\mathcal{L}\{\sin 2t\}$$

$$= \mathcal{L}\{\tfrac{1}{3}\sin 2t\}$$