Laplace Transform §6.1

Power Series

$$\sum_{n=0}^{\infty} a_n x^n = A(x)$$

$$\sum_{n=0}^{\infty} a(n)\, x^n = A(x) \quad , \qquad a(n) = a_n \, , \quad n = 0, 1, 2, 3, \ldots$$

<u>Ex</u>: $a(n) = 1 \iff \displaystyle\sum_{n=0}^{\infty} x^n = 1 + x + x^2 + x^3 + \cdots$

$$= A(x) = \frac{1}{1-x} \, , \quad \text{for } |x| < 1$$

$a(n) = \dfrac{1}{n!} \iff \displaystyle\sum_{n=0}^{\infty} \frac{x^n}{n!} = 1 + x + \frac{x^2}{2} + \frac{x^3}{3!} + \cdots$

$$A(x) = e^x \, , \qquad \text{for all } x \in \mathbb{R}$$

<u>Continuous analogue</u>:

$n = 0, 1, 2, 3, \ldots \quad \rightsquigarrow \quad t \geq 0$

$a(n) \quad \rightsquigarrow \quad f(t)$

$\displaystyle\sum_{n=0}^{\infty} \quad \rightsquigarrow \quad \int_0^{\infty}$

$\displaystyle\sum_{n=0}^{\infty} a_n x^n = A(x) \quad \rightsquigarrow \quad \int_0^{\infty} f(t)\, x^t \, dt = F(x)$

$$x^t = (e^{\ln x})^t = e^{(\ln x)t} = e^{-st} \qquad \ln x = -s$$

Laplace Transform

$$\boxed{\int_0^\infty f(t)e^{-st}\,dt = F(s) = \mathcal{L}\{f(t)\} \quad \text{for all } s \text{ s.t. the integral is defined}}$$

<u>Aside</u> : Fourier transform

$$f(t) \rightsquigarrow F(\omega) = \int_{-\infty}^{\infty} f(t)e^{-i\omega t}$$

t = time $\qquad$ ω = frequency

<u>Applications of Laplace transform</u>

* Solving diff. eqn's

* Probability :

$\mathcal{L}\{$probability distribution function$\}$

= moment generating function

* Statistical Physics

$\mathcal{L}\{$density of state function$(E)\}$

= partition function $\left(\frac{1}{KT}\right)$

Ex $\mathcal{L}\{1\} = \int_0^\infty e^{-st}\,dt$

$$= \lim_{A\to\infty} \int_0^A e^{-st}\,dt$$

$$= \lim_{A\to\infty} \left(-\frac{1}{s} e^{-st} \Big|_0^A \right) \quad \text{if } s \neq 0$$

$$= \lim_{A\to\infty} \left(-\frac{1}{s} e^{-sA} + \frac{1}{s} \right)$$

$$= \begin{cases} \frac{1}{s}, & s > 0 \\ \text{diverges}, & s \leq 0 \end{cases}$$

(Note if $s = 0$, $\int_0^\infty e^{0t}\,dt = \int_0^\infty 1\,dt$ diverges)

$$\boxed{\mathcal{L}\{1\} = F(s) = \frac{1}{s}} \quad \text{if } s > 0$$

Ex $\mathcal{L}\{e^{at}\} = \int_0^\infty e^{at} e^{-st}\,dt$

$$= \int_0^\infty e^{-(s-a)t}\,dt$$

$$= \frac{1}{s-a} \quad \text{if } s - a > 0$$

i.e. $s > a$

Ex $\mathcal{L}\{\sin(bt)\}$

$$= F(s) = \int_0^\infty \sin(bt)\, e^{-st}\, dt \quad , \quad s>0$$

i.b.p

$$= -\frac{1}{b}\cos(bt)\, e^{-st}\Big|_0^\infty - \frac{s}{b}\int_0^\infty \cos(bt)\, e^{-st}\, dt$$

i.b.p.

$$= \frac{1}{b} - \frac{s}{b}\left(\cancelto{0}{\frac{1}{b}\sin(bt)\, e^{-st}\Big|_0^\infty} + \frac{s}{b}\int_0^\infty \sin(bt)\, e^{-st}\, dt \right)$$

$$= \frac{1}{b} - \frac{s^2}{b^2}\int_0^\infty \sin(bt)\, e^{-st}\, dt$$

$$F(s) = \frac{1}{b} - \frac{s^2}{b^2} F(s)$$

$$\boxed{\mathcal{L}\{\sin(bt)\} = F(s) = \frac{b}{s^2+b^2}} \quad , \quad s>0$$