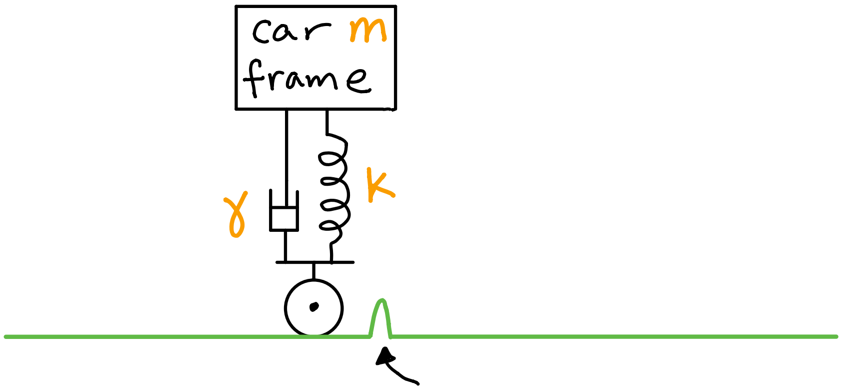
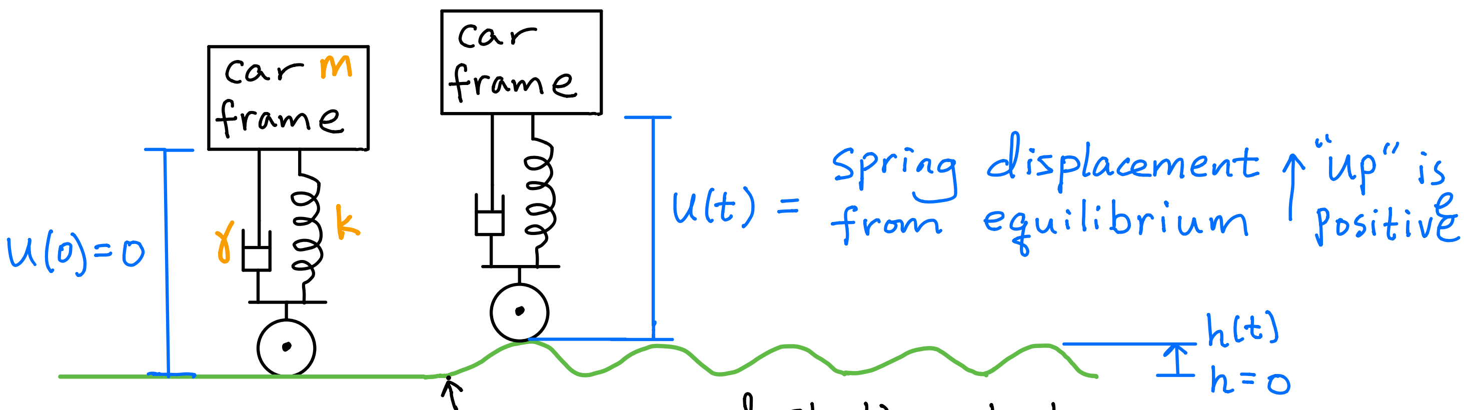
Forced Vibrations §3.8



sudden bump at $t=0$ gives nonzero initial condition

$$mu'' = -ku - \gamma u' \qquad (u = \text{displacement from equil.})$$

$$mu'' + \gamma u' + ku = 0$$



$$m(\text{car frame acceleration}) = -ku - \gamma u'$$

$$m(u+h)'' = -ku - \gamma u'$$

$$mu'' + \gamma u' + ku = -mh''$$

nonhomg. term: effective external driving force

There are many other examples of forced vibrations (e.g. anytime sound wave hits an object)

$$mu'' + \gamma u' + ku = \underbrace{F(t)}$$

consider periodic driving force

$$F(t) = F_0 \cos(\omega t) \qquad F_0, \omega > 0$$

constants

$$\text{or } F_0 \sin(\omega t)$$

Undamped Forced Vibration

$$mu'' + ku = F_0 \cos(\omega t)$$

Gen soln

$$\boxed{u(t) = C_1 \cos(\omega_0 t) + C_2 \sin(\omega_0 t) + u_p(t)} \qquad \omega_0 = \sqrt{\frac{k}{m}}$$

$\boxed{\text{Case 1}: \omega \neq \omega_0}$ (Beats)

Ansatz: $u_p(t) = A\cos(\omega t) + B\sin(\omega t)$

$$F_0 \cos(\omega t) = m u_p'' + k u_p$$

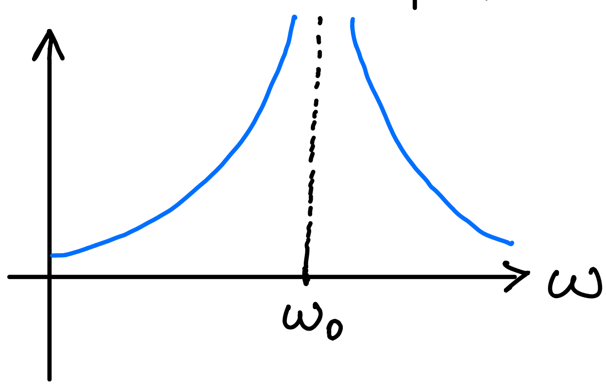
$$= m\left(-A\omega^2\cos(\omega t) - B\omega^2 \sin(\omega t)\right) + k\left(A\cos(\omega t) + B\sin(\omega t)\right)$$

$$= \left(-m\omega^2 + k\right)A\cos(\omega t) + \left(-m\omega^2 + k\right)B\sin(\omega t)$$

$$\Rightarrow A = \frac{F_0}{-m\omega^2 + k} = \frac{F_0}{m(\omega_0^2 - \omega^2)}, \qquad B = 0$$

$$\text{Gen soln: } \boxed{u(t) = \underbrace{\overbrace{C_1\cos(\omega_0 t) + C_2\sin(\omega_0 t)}^{C\cos(\omega_0 t - \delta)} + \frac{F_0}{m(\omega_0^2-\omega^2)}\cos(\omega t)}_{\text{sum of waves of different frequencies}}}$$

$$\text{Amplitude of } u_p(t) = \left|\frac{F_0}{m(\omega_0^2-\omega^2)}\right| \to \infty \text{ when } \omega \to \omega^0$$



Useful Trig identities:

$$\sin\theta \pm \sin\varphi = 2\sin\frac{\theta\pm\varphi}{2}\cos\frac{\theta\mp\varphi}{2}$$

$$\cos\theta - \cos\varphi = -2\sin\frac{\theta+\varphi}{2}\sin\frac{\theta-\varphi}{2}$$

$$\cos\theta + \cos\varphi = 2\cos\frac{\theta+\varphi}{2}\cos\frac{\theta-\varphi}{2}$$

Example: Let $u(0) = 0$, $u'(0) = 0$

$$\Rightarrow C_1 = -\frac{F_0}{m(\omega_0^2 - \omega^2)}, \quad C_2 = 0$$

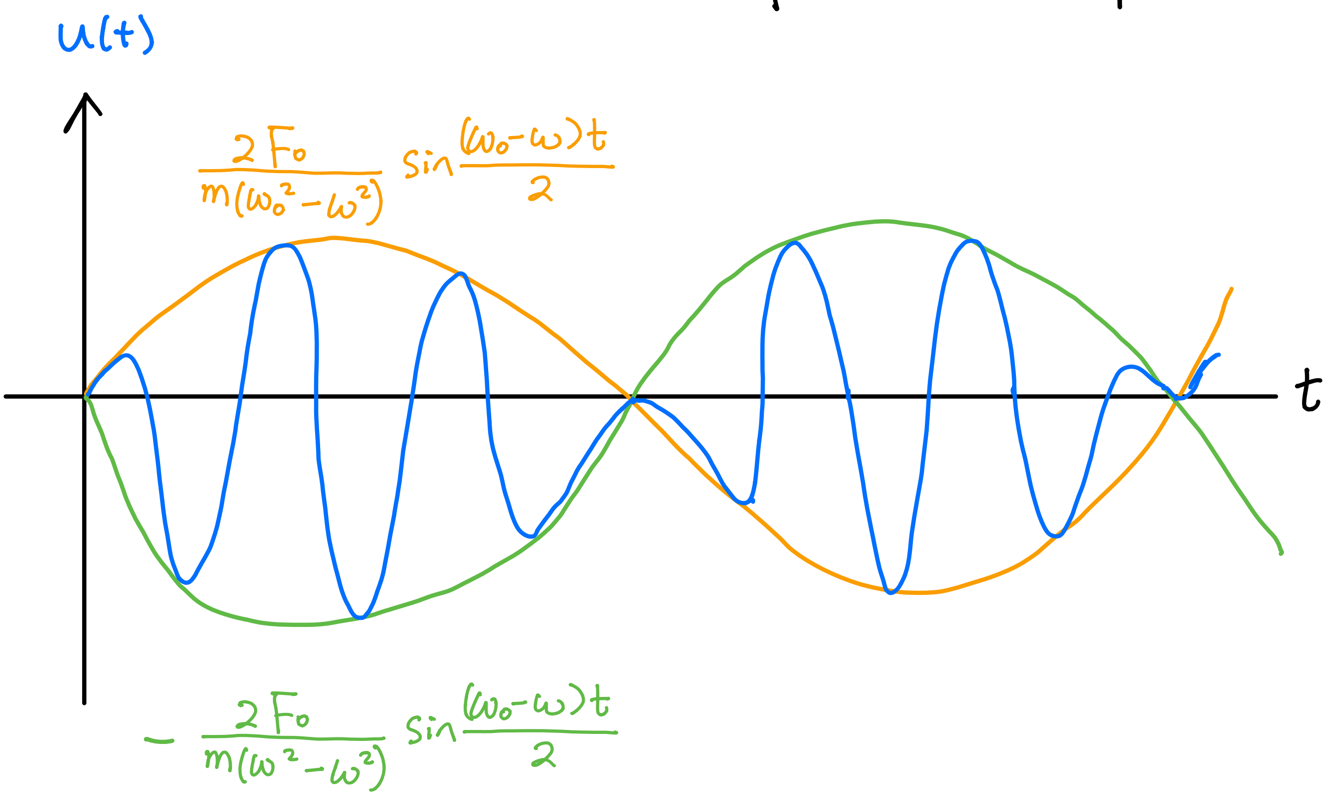
$$\Rightarrow u(t) = \frac{F_0}{m(\omega_0^2 - \omega^2)}\left(\cos(\omega t) - \cos(\omega_0 t)\right)$$

$$= \frac{2F_0}{m(\omega_0^2 - \omega^2)} \sin\frac{(\omega_0 - \omega)t}{2} \sin\frac{(\omega_0 + \omega)t}{2}$$

$$\left|\frac{\omega_0 - \omega}{2}\right| < \frac{\omega_0 + \omega}{2}$$

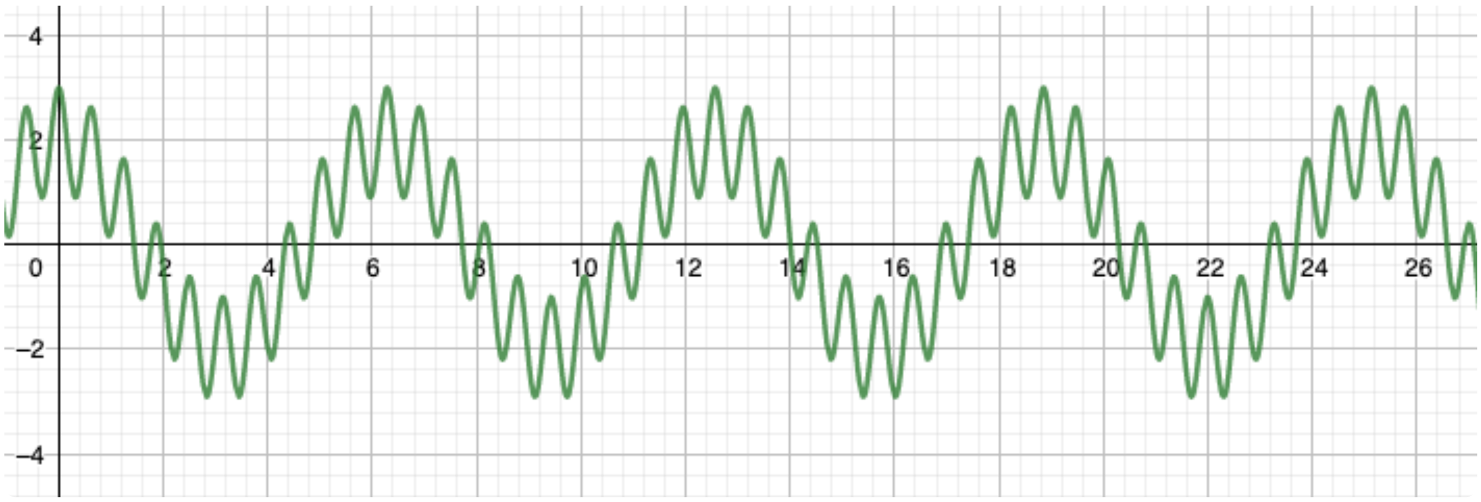
$$\Rightarrow \text{treat } \left|\frac{2F_0}{m(\omega_0^2 - \omega^2)} \sin\frac{(\omega_0 - \omega)t}{2}\right|$$

as an amplitude envelope



Amplitude modulation

Example: $2\cos t + \cos(10t)$



Example: $u'' + 25u = \cos 3t$, $u(0) = 0$, $u'(0) = 0$

char eq: $r^2 + 25 = 0$

$$r^2 = -25$$

$$r = \pm 5i$$

$$u(t) = C_1 \cos 5t + C_2 \sin 5t + u_p(t)$$

Ansatz: $u_p(t) = A\cos 3t + B\sin 3t$

$$u_p''(t) = -9A\cos 3t - 9B\sin 3t$$

$$u_p'' + 25u_p = \cos 3t$$

$$(-9A\cos 3t - 9B\sin 3t) + 25(A\cos 3t + B\sin 3t) = \cos 3t$$

$$16A\cos 3t + 16B\sin 3t = \cos 3t$$

$16A = 1$, $16B = 0$

$A = \frac{1}{16}$, $B = 0$

$$u(t) = c_1 \cos 5t + c_2 \sin 5t + \frac{1}{16} \cos 3t$$

$$0 = u(0) = c_1 + \frac{1}{16} \quad \Rightarrow \quad c_1 = -\frac{1}{16}$$

$$u'(t) = -5c_1 \sin 5t + 5c_2 \cos 5t - \frac{3}{16} \sin 3t$$

$$0 = u'(0) = 5c_2 \quad \Rightarrow \quad c_2 = 0$$

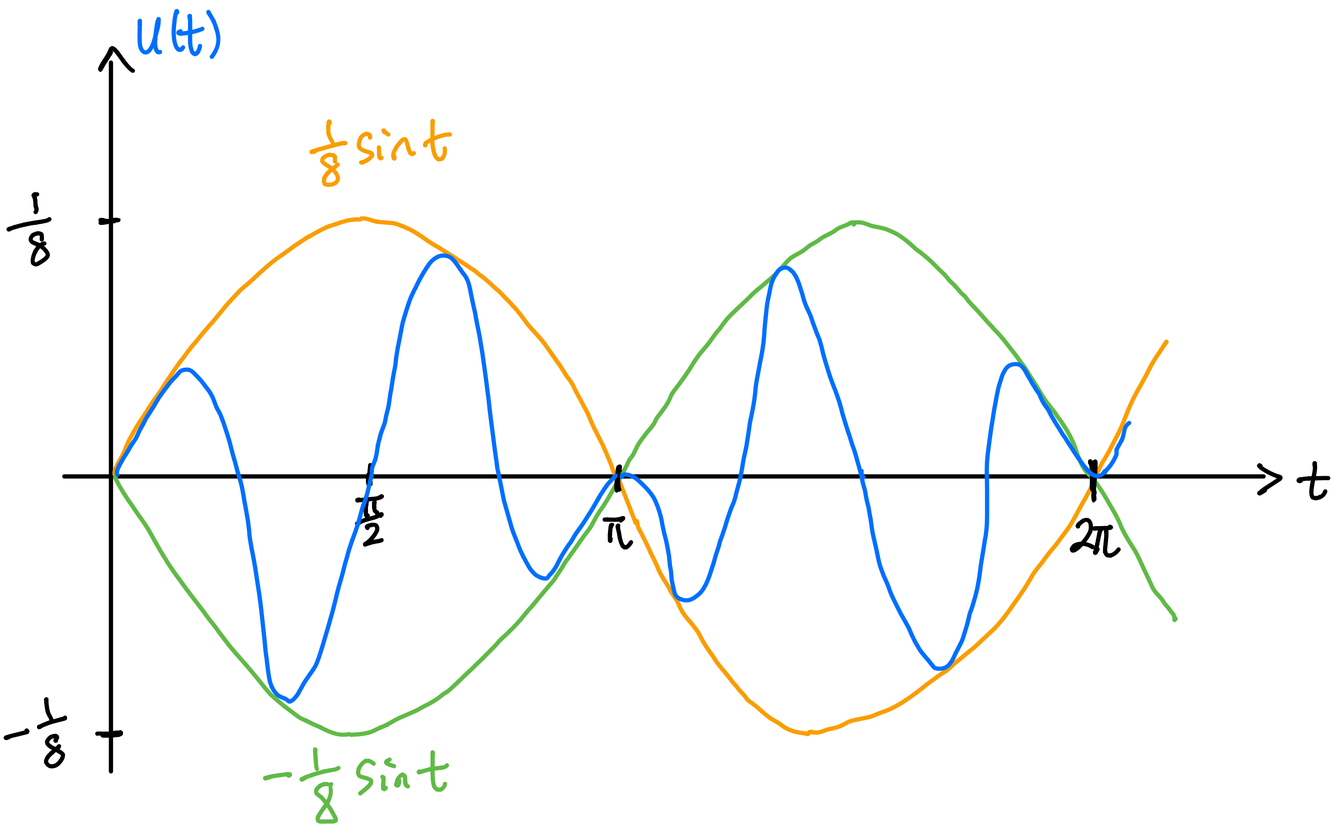
$$u(t) = -\frac{1}{16} \cos 5t + \frac{1}{16} \cos 3t$$

$$= -\frac{1}{16} (\cos 5t - \cos 3t)$$

$$= \frac{1}{8} \sin \frac{(5-3)t}{2} \sin \frac{(5+3)t}{2}$$

$$u(t) = \underbrace{\frac{1}{8} \sin t}_{\text{Amplitude}} \sin 4t$$

Period of $\sin 4t = \frac{2\pi}{4} = \frac{\pi}{2}$



$\boxed{\text{Case 2}: \ \omega = \omega_0}$ (Resonance)

$$mu'' + ku = F\cos(\omega_0 t)$$

<u>Ansatz</u>: $u_p(t) = At\cos(\omega_0 t) + Bt\sin(\omega_0 t)$

$$F_0\cos(\omega_0 t) = mu_p'' + ku_p$$

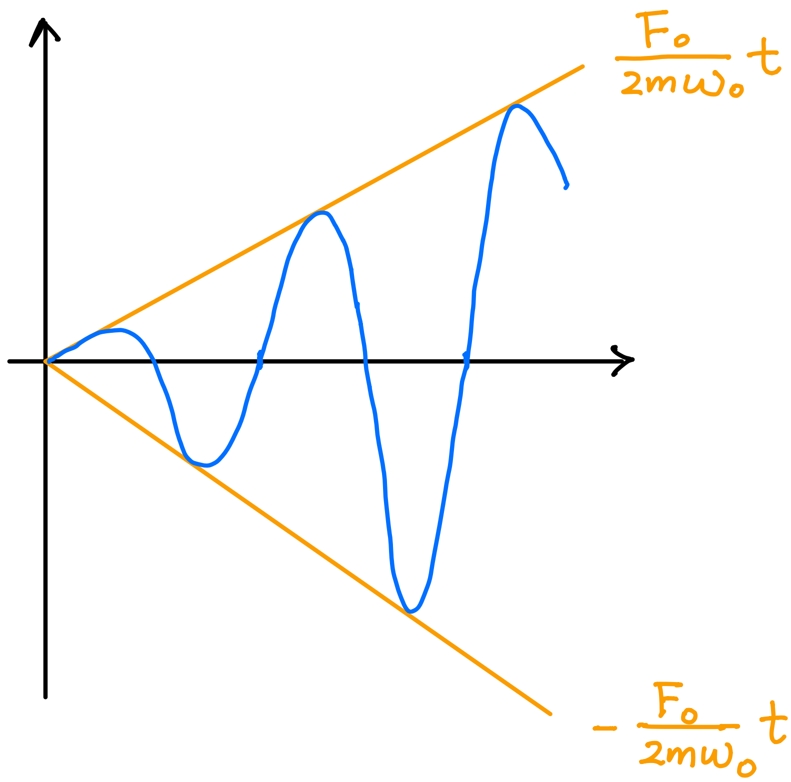
$$= m(u_p'' + \omega_0^2 u_p)$$

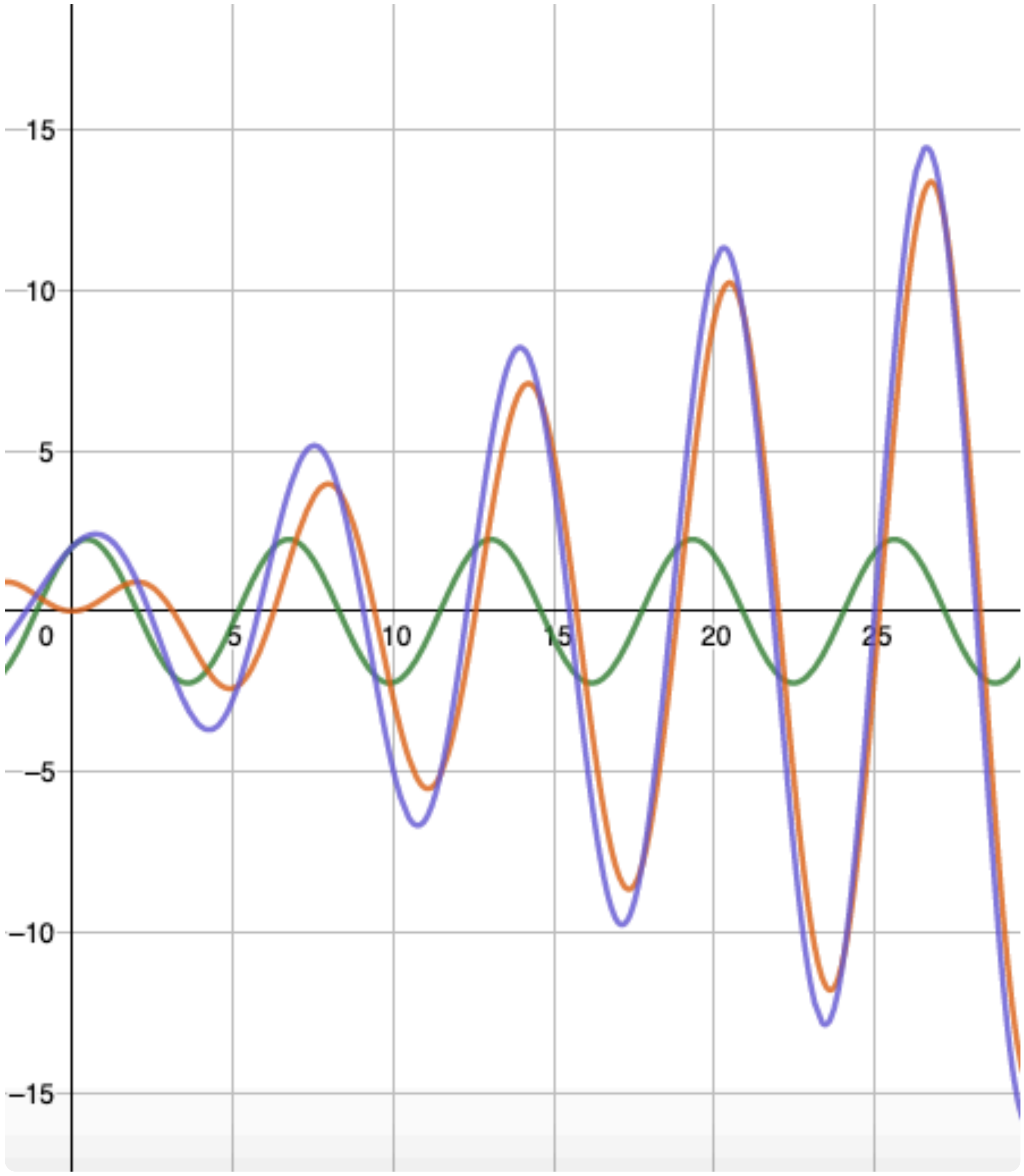
After simplification → $= m(-2A\omega_0 \sin(\omega_0 t) + 2B\omega_0\cos(\omega_0 t))$

$$\Rightarrow A = 0, \qquad B = \frac{F_0}{2m\omega_0}$$

Gen soln: $\boxed{u(t) = \underbrace{C_1\cos(\omega_0 t) + C_2\sin(\omega_0 t)}_{= C\cos(\omega_0 t - \delta)} + \frac{F_0}{2m\omega_0} t\sin(\omega_0 t)}$

$$u_p(t) = \frac{F_0}{2m\omega_0} t\sin(\omega_0 t)$$





Purple : $2\cos t + \sin t + \frac{1}{2} t \sin t$

Orange: $\frac{1}{2} t \sin t$

Green: $2\cos t + \sin t$

Periodic driving force + damping

$$mu'' + \gamma u' + ku = F_0 \cos(\omega t)$$

Gen soln : $u(t) = \underbrace{c_1 u_1(t) + c_2 u_2(t)} + u_p(t)$

Gen soln to $mu'' + \gamma u' + ku = 0$
always decays, called "transient solution"

Ansatz: $u_p(t) = C\cos\omega t + D\sin\omega t$

$u_p(t)$ is called the "steady state soln"

$$u_p'(t) = -\omega C\sin\omega t + \omega D\cos\omega t$$

$$u_p''(t) = -\omega^2 C\cos\omega t - \omega^2 D\sin\omega t$$

$$F_0\cos\omega t = mu_p'' + \gamma u_p' + ku_p$$

$$= \left[(k-m\omega^2)C + \gamma\omega D\right]\cos\omega t + \left[(k-m\omega^2)D - \gamma\omega C\right]\sin\omega t$$

$$\Rightarrow \left.\begin{aligned} (k-m\omega^2)C + \gamma\omega D &= F_0 \\ -\gamma\omega C + (k-mc^2)D &= 0 \end{aligned}\right\}$$

$$C = \frac{(k-m\omega^2)F_0}{(k-m\omega^2)^2 + (\gamma\omega)^2} = \frac{m(\omega_0^2-\omega^2)F_0}{m^2(\omega_0^2-\omega^2)^2 + (\gamma\omega)^2}$$

$$D = \frac{\gamma\omega F_0}{(k-m\omega^2)^2 + (\gamma\omega)^2} = \frac{\gamma\omega F_0}{m^2(\omega_0^2-\omega^2)^2 + (\gamma\omega)^2}$$

Write $u_p = C\cos\omega t + D\sin\omega t = A\cos(\omega t - \delta)$

$$A^2 = C^2 + D^2 = \frac{m^2(\omega_0^2-\omega^2)^2 F_0^2 + (\gamma\omega)^2 F_0^2}{\left(m^2(\omega_0^2-\omega^2)^2 + (\gamma\omega)^2\right)^2}$$

$$= \frac{F_0^2}{(k-m\omega^2)^2 + (\gamma\omega)^2}$$

$$= \frac{F_0^2}{m^2(\omega_0^2-\omega^2)^2 + (\gamma\omega)^2}$$

Amplitude $A = \dfrac{F_0}{\sqrt{m^2(\omega_0^2-\omega^2)^2 + (\gamma\omega)^2}}$

denominator always $\neq 0$ if $\gamma \neq 0$

<u>Remark</u> This calculation can also be done by

Ansatz: $u_p = A\cos(\omega t - \delta) = \mathrm{Re}\,\underbrace{Ae^{i(\omega t-\delta)}}_{=\tilde{u}_p}$

$$\mathrm{Re}\left(F_0 e^{i\omega t}\right) = \mathrm{Re}\left(m\tilde{u}_p'' + \gamma\tilde{u}_p' + k\tilde{u}_p\right)$$

<u>Plot A vs ω</u>

$$A(\omega) = \frac{F_0}{\sqrt{m^2(\omega_0^2-\omega^2)^2 + (\gamma\omega)^2}}$$

$\omega \to 0 \quad \Rightarrow \quad A \to \dfrac{F_0}{k}$

$\omega \to \infty \quad \Rightarrow \quad A \to 0$

A has a max when $m^2(\omega_0^2-\omega^2)^2+(\gamma\omega)^2$ has a min

$$0=\frac{d}{d\omega}\left[m^2(\omega_0^2-\omega^2)^2+(\gamma\omega)^2\right]$$

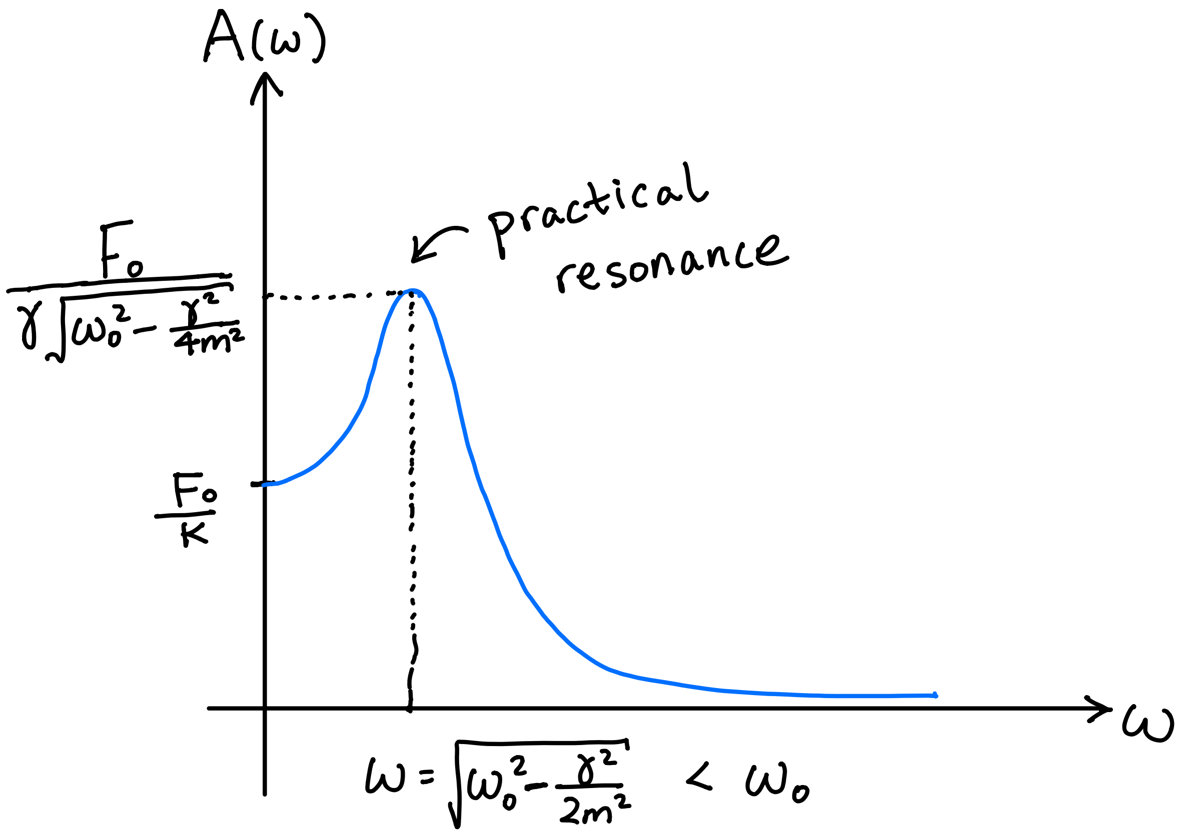
$$=\left(-2m^2(\omega_0^2-\omega^2)+\gamma^2\right)2\omega$$

$$\omega=0 \quad , \quad \omega=\sqrt{\omega_0^2-\frac{\gamma^2}{2m^2}}$$

real when $\omega_0^2-\frac{\gamma^2}{2m^2}\geq 0$

$$\Rightarrow \omega_0^2-\left(\frac{\gamma}{2m}\right)^2>\omega_0^2-\frac{\gamma^2}{2m^2}\geq 0$$

$$\left(\begin{matrix} mu''+\gamma u'+ku=0 \\ \text{underdamped}\end{matrix}\right)$$



By adjusting damping γ, can change resonance frequency.

Ex: $u'' + 4u' + 13u = 20\cos(\omega t)$

char eq: $r^2 + 4r + 13 = 0$

$$r = -2 \pm 3i$$

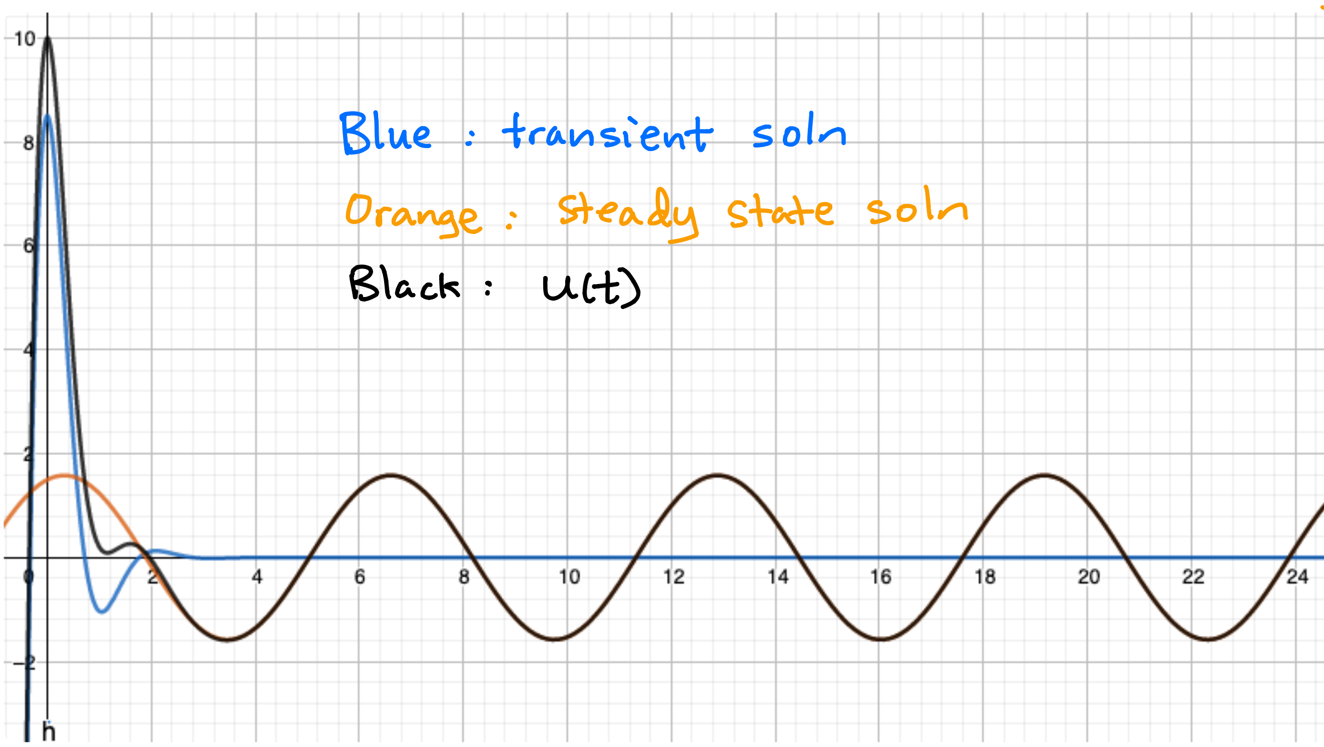
$$u = c_1 e^{-2t}\cos(3t) + c_2 e^{-2t}\sin(3t) + U_p(t)$$

Resonance happens when

$$\omega = \sqrt{13 - \frac{4^2}{2}} = \sqrt{5} \approx 2.24$$

(a) $u'' + 4u' + 13u = 20\cos(t)$, $u(0) = 10$, $u'(0) = 0$

$$u(t) = 8.5e^{-2t}\cos(3t) + 5.5e^{-2t}\sin(3t) + \underbrace{\frac{3}{2}\cos t + \frac{1}{2}\sin t}_{=1.6\cos(t-0.3)}$$



Driving forces with different freq's
but the same amplitude

(b) $u'' + 4u' + 13u = \overbrace{20\cos(t) + 20\cos(13t)}$

$$u_p(t) = \underbrace{\left(\frac{3}{2}\cos t + \frac{1}{2}\sin t\right)}_{=1.6\cos(t-0.3)} + \underbrace{\left(-\frac{3}{26}\cos(13t) + \frac{1}{26}\sin(13t)\right)}_{=0.1\cos(13t-3.4)}$$