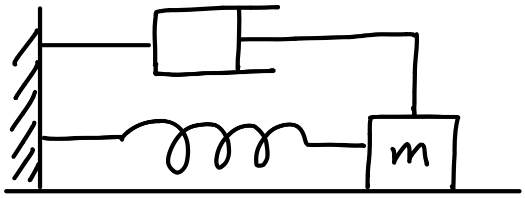
Damped Harmonic Oscillator (§3.7)



damper resists motion

(hydraulic device: piston pushing fluid)

$$mu'' = -ku \underbrace{-\gamma u'}_{\text{damping}}$$

Photo credit: Wikipedia "Shock absorber"

$$mu'' + \gamma u' + ku = 0$$

char eq: $mr^2 + \gamma r + ku = 0$

roots: $r = -\dfrac{\gamma \pm \sqrt{\gamma^2 - 4mk}}{2m}$

$$= -\frac{\gamma}{2m} \pm \sqrt{\left(\frac{\gamma}{2m}\right)^2 - \frac{k}{m}}$$

$\left(\frac{\gamma}{2m}\right)^2 - \frac{k}{m} > 0$, distinct real roots (Overdamped)

$\left(\frac{\gamma}{2m}\right)^2 - \frac{k}{m} = 0$, repeated real roots (Critically damped)

} real exp soln

$\left(\frac{\gamma}{2m}\right)^2 - \frac{k}{m} < 0$, cx roots } oscillation (Underdamped)

Underdamped

Cx roots: $r = -\frac{\gamma}{2m} \pm \sqrt{-\left(\frac{k}{m} - \left(\frac{\gamma}{2m}\right)^2\right)}$

$$= -\frac{\gamma}{2m} \pm i\mu$$

$$\mu = \sqrt{\frac{k}{m} - \left(\frac{\gamma}{2m}\right)^2}$$

Gen soln:

$$u(t) = c_1 e^{-\frac{\gamma}{2m}t}\cos(\mu t) + c_2 e^{-\frac{\gamma}{2m}t}\sin(\mu t)$$

$$= e^{-\frac{\gamma}{2m}t}\left(c_1\cos(\mu t) + c_2\sin(\mu t)\right)$$

$$= e^{-\frac{\gamma}{2m}t} A\cos(\mu t - \delta)$$

Quasi-freq: $\mu = \sqrt{\frac{k}{m} - \left(\frac{\gamma}{2m}\right)^2} < \omega_0 = \sqrt{\frac{k}{m}}$

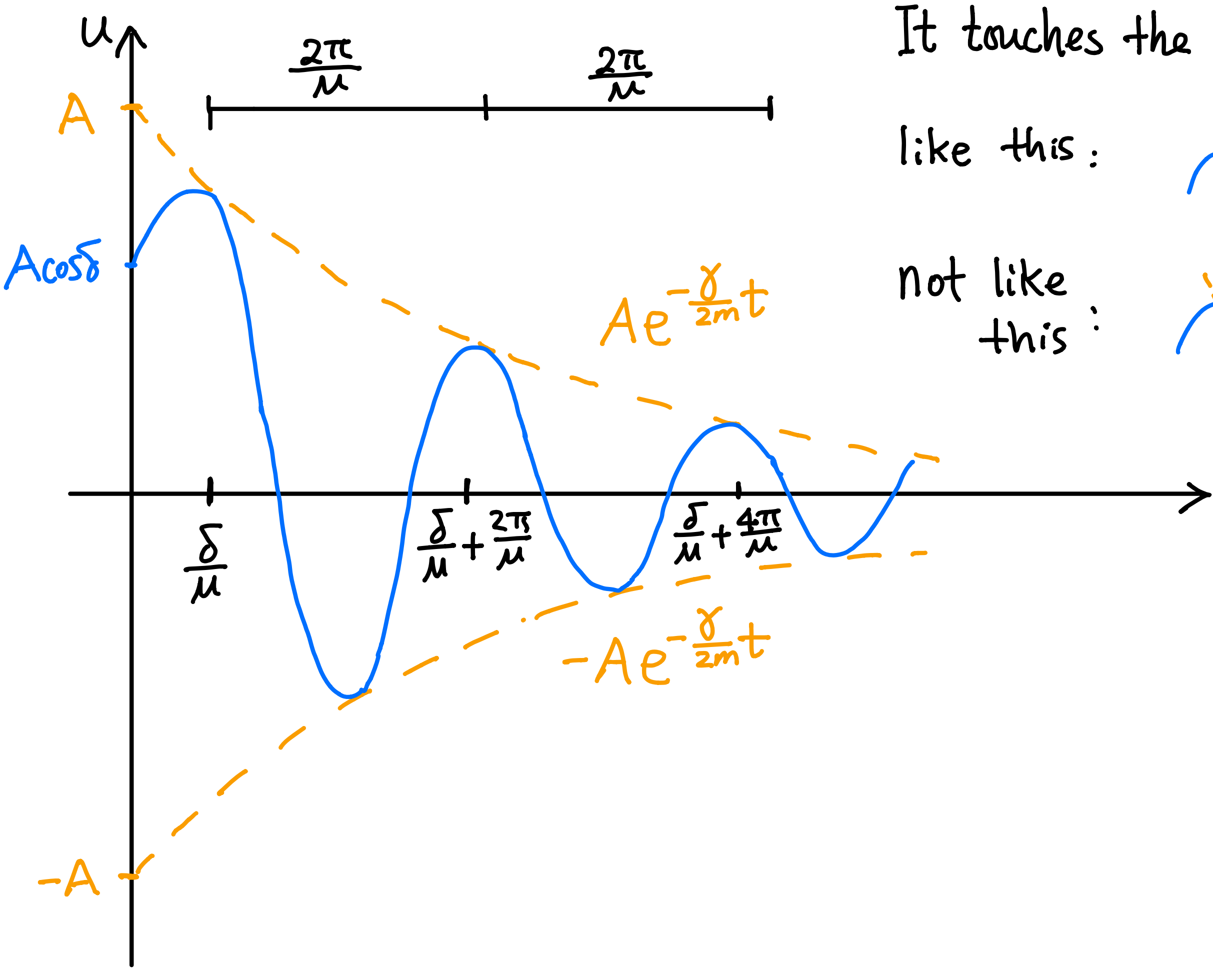
Quasi-Period: $T = \frac{2\pi}{\mu} > T_0 = \frac{2\pi}{\omega_0}$

Amplitude envelope: $\pm A e^{-\frac{\gamma}{2m}t}$

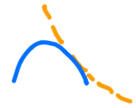
First time touching the upper envelope $Ae^{-\frac{\gamma}{2m}t}$:

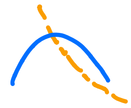
$\mu t - \delta = 0, \quad t = \frac{\delta}{\mu}$

(assuming we chose $-2\pi < \delta < 2\pi$, which we did, so the smallest $t>0$ s.t. $\mu t - \delta = 2\pi n$ happen when $\mu t - \delta = 0$)



It touches the envelope

like this: 

not like this: 

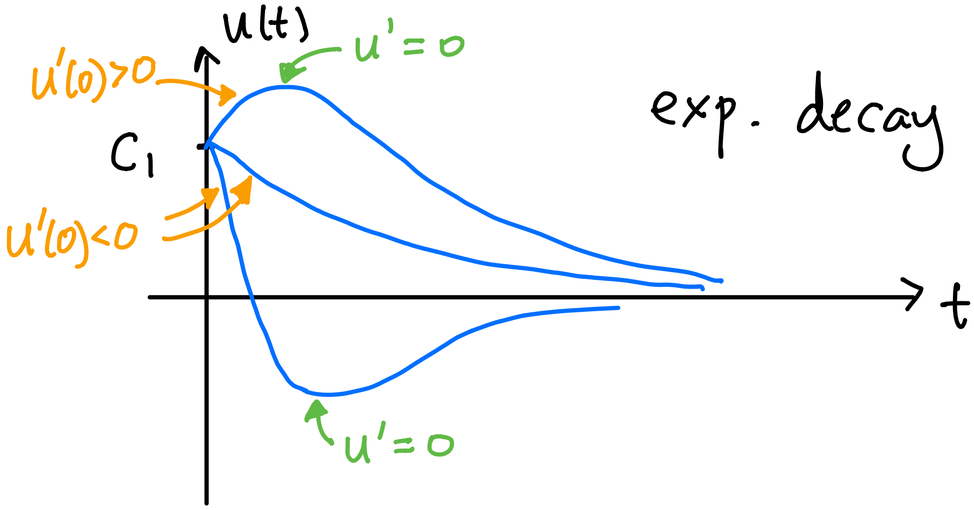
<u>Exercise</u>: At any time t_p s.t. $\cos(\mu t_p - \delta) = \pm 1$,

check that $u'(t_p) \neq 0$

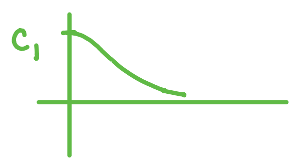
<u>Critically damped</u>: $\mu = 0$

<u>Gen soln</u>: $u(t) = c_1 e^{-\frac{\gamma}{2m}t} + c_2 t e^{-\frac{\gamma}{2m}t}$

$= e^{-\frac{\gamma}{2m}t}(c_1 + c_2 t)$



Comment added after lecture: forgot to mention $u'(0)$ could also be 0, like



* <u>Exercise</u>: check $u'(0) = C_2 - \frac{\gamma}{2m} C_1$

* Can use $u'(t) = 0$ to figure out the peak or bottom of the bump. (local max/min)

<u>Overdamped</u>:

real roots: $r = -\frac{\gamma}{2m} \pm \sqrt{\left(\frac{\gamma}{2m}\right)^2 - \frac{k}{m}}$

<u>Gen soln</u>:

$$u(t) = C_1 e^{\left(-\frac{\gamma}{2m} + \sqrt{\left(\frac{\gamma}{2m}\right)^2 - \frac{k}{m}}\right)t} + C_2 e^{\left(-\frac{\gamma}{2m} - \sqrt{\left(\frac{\gamma}{2m}\right)^2 - \frac{k}{m}}\right)t}$$

both terms are exp. decay

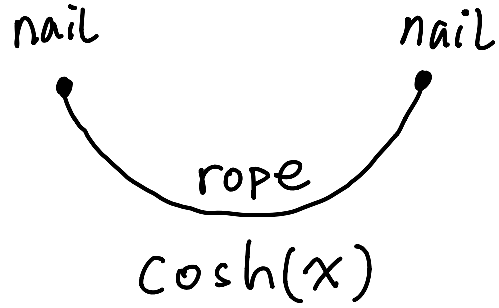
$$\frac{\gamma}{2m} > \sqrt{\left(\frac{\gamma}{2m}\right)^2 - \frac{k}{m}}$$

The graph looks "exactly" like the critically damped case

$$u(t) = e^{-\frac{\gamma}{2m}t}\left(C_1 e^{\sqrt{\left(\frac{\gamma}{2m}\right)^2 - \frac{k}{m}}\, t} + C_2 e^{-\sqrt{\left(\frac{\gamma}{2m}\right)^2 - \frac{k}{m}}\, t}\right)$$

$$= e^{-\frac{\gamma}{2m}t}\left(A\cosh\left(\sqrt{\left(\frac{\gamma}{2m}\right)^2 - \frac{k}{m}}\, t\right) + B\sinh\left(\sqrt{\left(\frac{\gamma}{2m}\right)^2 - \frac{k}{m}}\, t\right)\right)$$

Note about cosh, sinh



$$\cosh x = \frac{e^x + e^{-x}}{2}$$

$$\sinh x = \frac{e^x - e^{-x}}{2}$$

Recall:

$$\cos x = \frac{e^{ix} + e^{-ix}}{2}$$

$$\sin x = \frac{e^{ix} - e^{-ix}}{2i}$$

Comments added after lecture (since someone asked):

Notice $\cosh x + \sinh x = e^x$

$\cosh x - \sinh x = e^{-x}$

So $c_1 e^x + c_2 e^{-x}$

$$= c_1(\cosh x + \sinh x) + c_2(\cosh x - \sinh x)$$

$$= (c_1 + c_2)\cosh x + (c_1 - c_2)\sinh x$$

$$= A\cosh x + B\sinh x$$