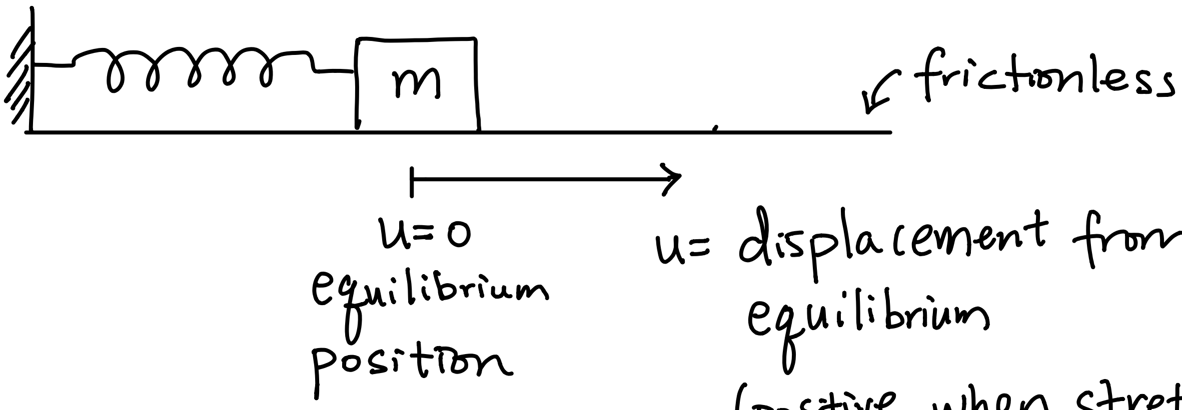
Harmonic Oscillator §3.7



u= displacement from equilibrium
(positive when stretched)

$$m u'' = F_{spring} = -ku \qquad \text{(Hooke's law)}$$

$$[k] = kg/s^2$$

$$\boxed{mu'' + ku = 0}$$

$$u'' + \frac{k}{m} u = 0$$

$$u'' + \omega_0^2 u = 0 \quad , \quad \omega_0 = \sqrt{\frac{k}{m}}$$

char eq: $r^2 + \omega_0^2 = 0$

$$r^2 = -\omega_0^2$$

$$r = \pm i\omega_0$$

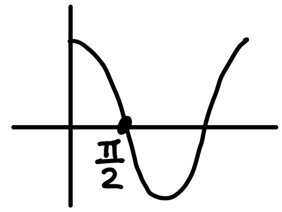
$$\boxed{u = C_1 \cos(\omega_0 t) + C_2 \sin(\omega_0 t)}$$

Experiment 1 to find k

Take $m = 1kg$, pull 1 unit to the right,

release from rest :

$$u(0) = 1, \quad u'(0) = 0$$



$$\xrightarrow{\text{solve IVP}} \quad u = \cos(\sqrt{k}\, t)$$

Then record how long it takes to arrive at equilibrium for the first time. Say it took $t_1 = 0.5$ sec.

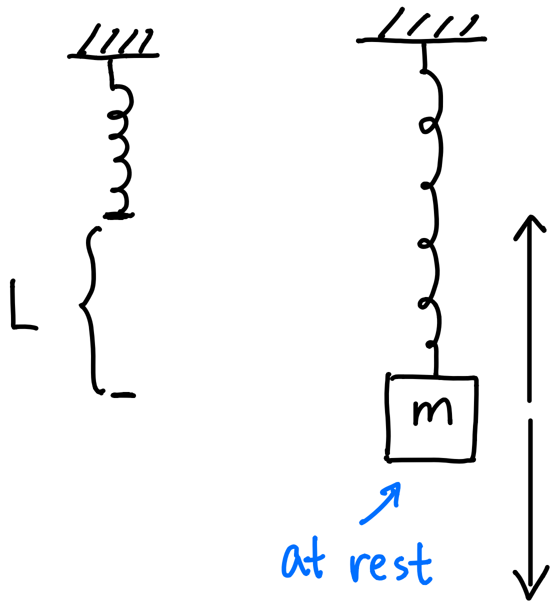
$$0 = u(0.5) = \cos(\sqrt{k}\ 0.5)$$

$$\sqrt{k}\ 0.5 = \frac{\pi}{2}$$

$$\sqrt{k} = \pi$$

$$k = \pi^2 \ \text{kg}/\text{s}^2$$

Experiment 2 for finding k



$$|F_{spring}| = kL$$

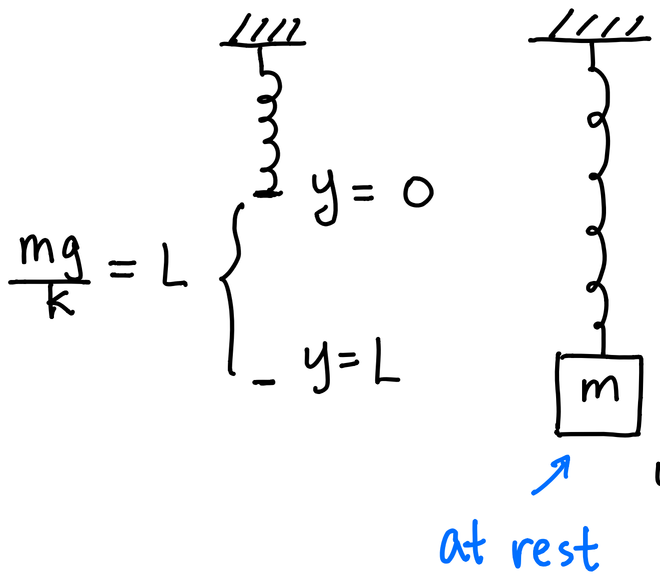
$$|F_{gravity}| = mg$$

$$kL = mg$$

$$\boxed{k = \frac{mg}{L}}$$

End of lecture 1

Hanging Harmonic Oscillator



$u = 0$ (downward "stretched" is positive)

$u = y - L$

$= y - \frac{mg}{k}$

$u(t) = 0$ is a constant/equilibrium soln

u = displacement from equilibrium

$mu'' =$ Force that causes motion/displacement

(i.e. F_{spring} in excess of gravity)

$= -ku$

$$mu'' + ku = 0$$

$$u = c_1 u_1 + c_2 u_2$$

$$\begin{cases} m\dfrac{d^2y}{dt} = F_{gravity} + F_{spring} = mg - ky \\ my'' + ky = mg \\ y = \underbrace{c_1y_1 + c_2y_2}_{\text{gen soln to } my''+ky=0} + \dfrac{mg}{k} \end{cases}$$

Compare y and u :

$$u = y - \frac{mg}{k} \quad , \quad \text{so} \quad y = u + \frac{mg}{k}$$

$$m\left(u + \frac{mg}{k}\right)'' + k\left(u + \frac{mg}{k}\right) = mg$$

$$mu'' + ku + \cancel{mg} = \cancel{mg}$$

$$mu'' + ku = 0$$

Notice : $mg - ky = 0 \Rightarrow y(t) = \frac{mg}{k}$ is an equilibrium soln

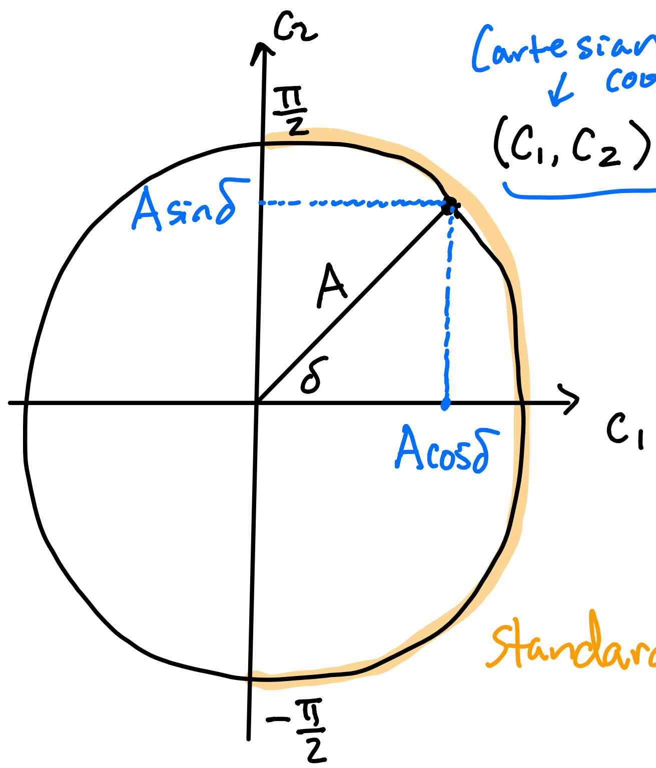
Harmonic Oscillator

$$\boxed{\begin{aligned} &m u'' + k u = 0 \\ &u = C_1 \cos(\omega_0 t) + C_2 \sin(\omega_0 t) \end{aligned}}$$

$\omega_0 = \sqrt{\frac{k}{m}}$

$$= \underbrace{A\cos\delta}_{=C_1} \cos(\omega_0 t) + \underbrace{A\sin\delta}_{=C_2} \sin(\omega_0 t)$$

$$\boxed{u(t) = A\cos(\omega_0 t - \delta)}$$



Cartesian coord.

(A, δ) = polar coord.

$$\underbrace{(C_1, C_2) = (A\cos\delta, A\sin\delta)}_{\text{determined by the initial condition}}$$

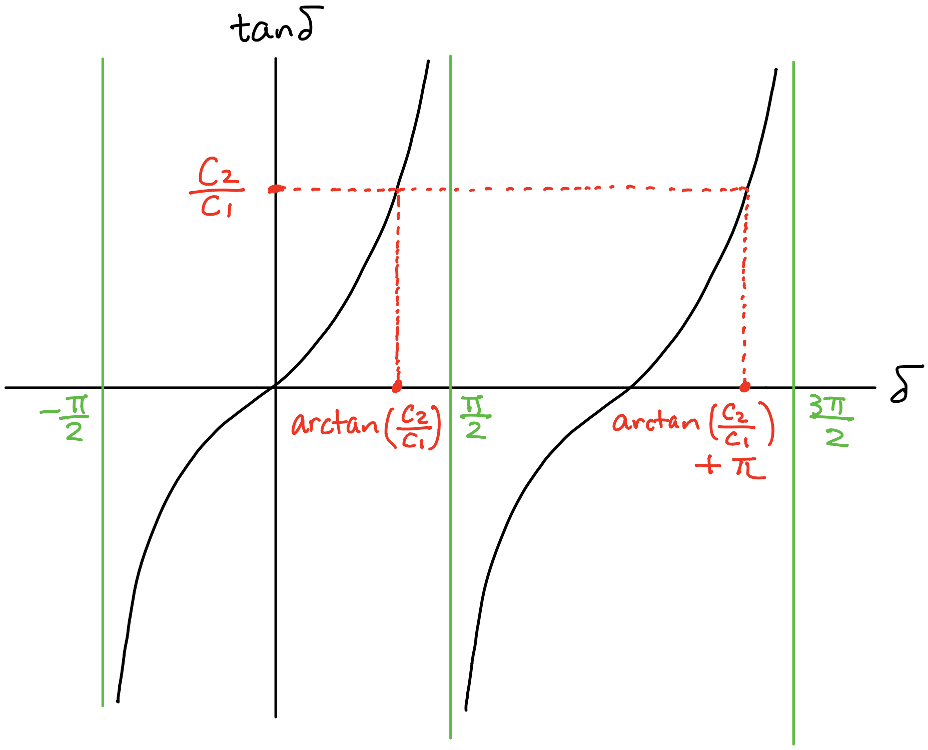
Standard arctan $\in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$

determined by initial conditions

Amplitude $A = \sqrt{c_1^2 + c_2^2} = \sqrt{A^2\cos^2\delta + A^2\sin^2\delta}$

Phase shift : $\left.\begin{aligned} \cos\delta &= \frac{C_1}{A} \\ \sin\delta &= \frac{C_2}{A} \end{aligned}\right\} \Rightarrow \underbrace{\tan\delta = \frac{C_2}{C_1}}$

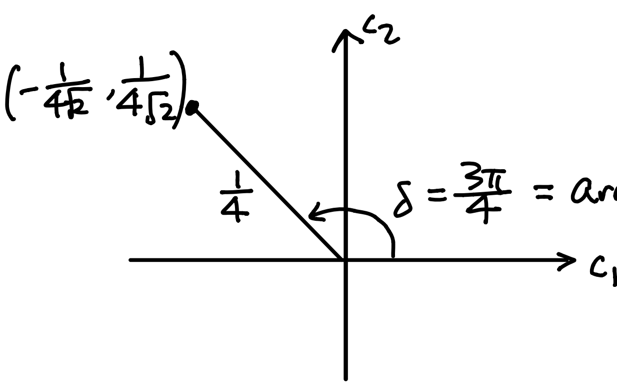
This carries less info than these. So need to also know the sign of C_1.



$$\begin{cases} \text{If } c_1 > 0, & \delta = \arctan\left(\frac{c_2}{c_1}\right) \\ \text{If } c_1 < 0, & \delta = \arctan\left(\frac{c_2}{c_1}\right) + \pi \\ \text{If } c_1 = 0, & \delta = \begin{cases} \frac{\pi}{2} \text{ if } c_2 > 0 \\ -\frac{\pi}{2} \text{ if } c_2 < 0 \end{cases} \end{cases}$$

__Ex1__: $u(t) = -\frac{1}{4\sqrt{2}} \cos(2\pi t) + \frac{1}{4\sqrt{2}} \sin(2\pi t)$

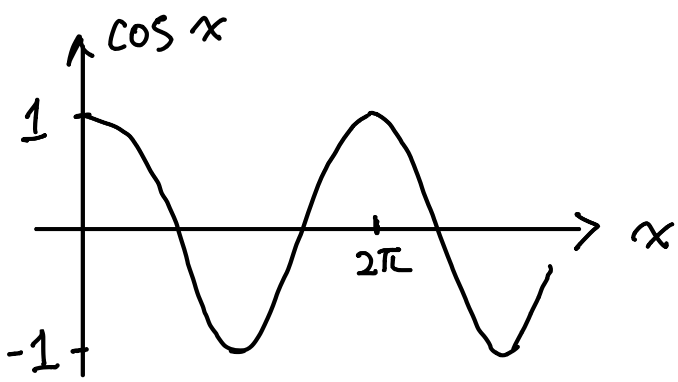
$$A = \sqrt{c_1^2 + c_2^2} = \sqrt{\frac{1}{32} + \frac{1}{32}} = \frac{1}{4}$$



$$\delta = \frac{3\pi}{4} = \arctan\left(\frac{c_2}{c_1}\right) + \pi = \arctan(-1) + \pi$$

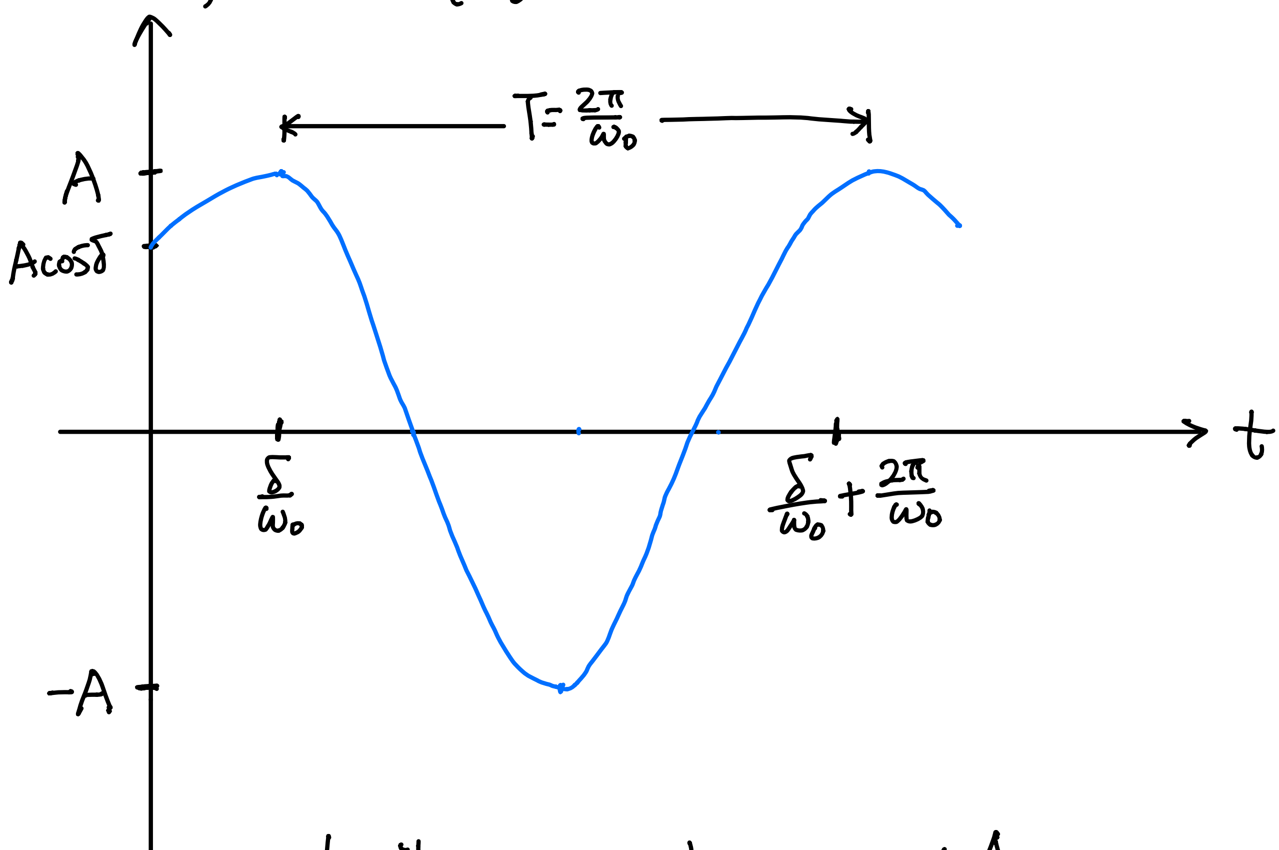
$$u(t) = \frac{1}{4} \cos\left(2\pi t - \frac{3\pi}{4}\right)$$

Graph $u(t) = A\cos(\omega_0 t - \delta)$



$$\cos(x) = \cos(x + 2\pi)$$

$$u(t) = A\cos(\omega_0 t - \delta)$$



velocity $= 0$ when $u = \pm A$

First peak happens when $\omega_0 t - \delta = 0$

i.e. $t = \frac{\delta}{\omega_0}$

Comment added after the lecture: this is assuming we chose $-2\pi < \delta < 2\pi$, so the smallest $t > 0$ s.t. $\omega_0 t - \delta = 2\pi n$ happens when $\omega_0 t - \delta = 0$

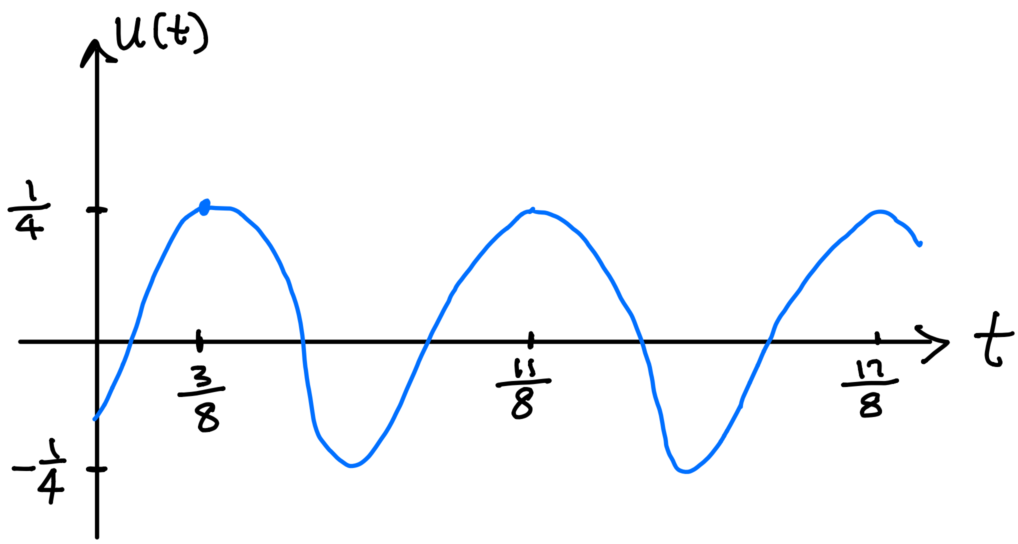
<u>Period</u>: $\cos(\omega_0 t - \delta) = \cos(\omega_0 t - \delta + 2\pi)$

$$= \cos\left(\omega_0\left(t + \frac{2\pi}{\omega_0}\right) - \delta\right)$$

$$T = \frac{2\pi}{\omega_0} = 2\pi\sqrt{\frac{m}{k}}$$

<u>Natural Frequency</u> $\omega_0 = \sqrt{\frac{k}{m}}$

<u>Ex 1 cont'd</u> Graph $u(t) = \frac{1}{4}\cos\left(2\pi t - \frac{3\pi}{4}\right)$



First peak when $2\pi t - \frac{3\pi}{4} = 0$

$$2\pi t = \frac{3\pi}{4}$$

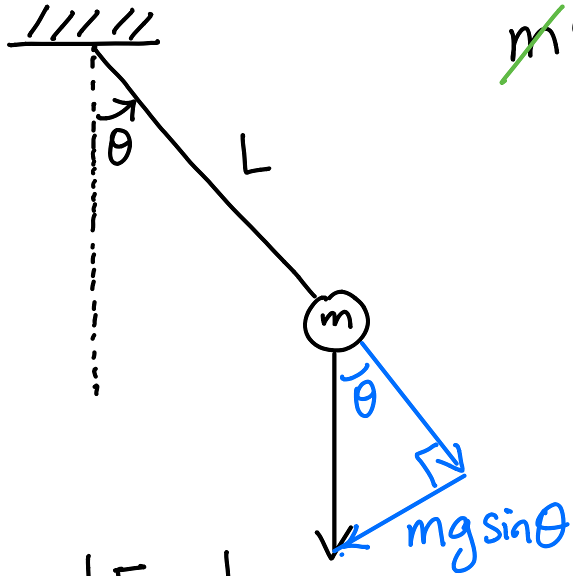
$$t = \frac{3}{8}$$

Period: $T = \frac{2\pi}{\omega_0} = \frac{2\pi}{2\pi} = \underline{1}$

There are many examples of vibration in nature.

End of lecture 2

Pendulum



$$|F_{grav}| = mg$$

$$\cancel{m}\frac{d^2(L\theta)}{dt^2} = -\cancel{m}g\sin\theta$$

$$L\frac{d^2\theta}{dt^2} = -g\sin\theta$$

$$\boxed{\theta'' + \frac{g}{L}\sin\theta = 0}$$ (nonlinear)

small oscillations $\Rightarrow \sin\theta \approx \theta$

$$\boxed{\theta'' + \frac{g}{L}\theta = 0}$$ (Linearization)