§ 3.5

Homog linear eqn

$$L[y] = y'' + p(t)\,y' + q(t)\,y = 0$$

General soln: $c_1 y_1(t) + c_2 y_2(t)$

Nonhomog linear eqn

$$L[y] = y'' + p(t)\,y' + q(t)\,y = g(t) \neq 0$$

Observe: If $Y_1(t)$, $Y_2(t)$ are soln's to a nonhomog. linear eqn,

$$L[Y_1] = g(t)$$

$$L[Y_2] = g(t)$$

$$\begin{aligned}
L[c_1 Y_1 + c_2 Y_2] &= (c_1 Y_1 + c_2 Y_2)'' + p(t)(c_1 Y_1 + c_2 Y_2)' + q(t)(c_1 Y_1 + c_2 Y_2) \\
&= c_1 (Y_1'' + p(t) Y_1' + q(t) Y_1) + c_2 (Y_2'' + p(t) Y_2' + q(t) Y_2) \\
&= c_1 L[Y_1] + c_2 L[Y_2] \\
&= c_1 g(t) + c_2 g(t) \neq g(t)
\end{aligned}$$

$$L[Y_1 - Y_2] = L[Y_1] - L[Y_2] = g(t) - g(t) = 0$$

$$Y_1 - Y_2 = c_1 y_1(t) + c_2 y_2(t)$$

Thm 3.5.2 The general soln to

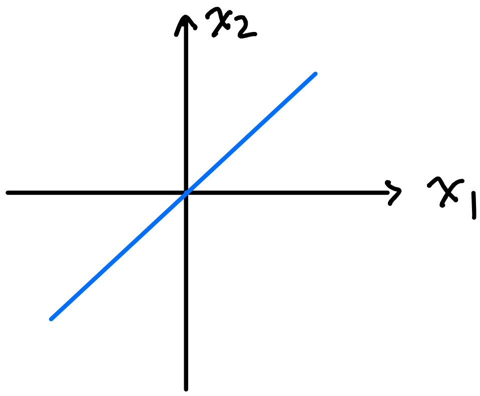
$$y'' + p(t)\, y' + q(t)\, y = g(t) \quad (\text{i.e. } L[y] = g(t))$$

is of the form

$$y = \underbrace{c_1 y_1 + c_2 y_2}_{\substack{\text{gen soln to} \\ L[y]=0}} + \underbrace{y_p(t)}_{\substack{\text{a particular soln to} \\ L[y]=g(t)}}$$

linear algebra analogue

$$a_1 x_1 + a_2 x_2 = 0$$



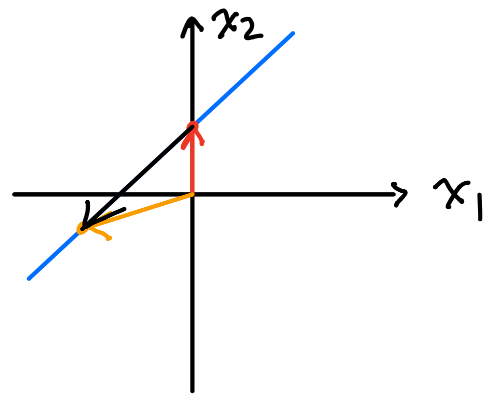
Soln: $\{(x_1, x_2)\}$

$$= \left\{ \left(x_1,\ x_2 = -\frac{a_1}{a_2} x_1\right) \right\}$$

$$= \left\{ \left(c,\ -\frac{a_1}{a_2} c\right) \right\}$$

$$= \left\{ c\left(1,\ -\frac{a_1}{a_2}\right) \right\}$$

$$a_1 x_1 + a_2 x_2 = b$$



Soln $\left\{ \left(x_1,\ x_2 = \frac{b}{a_2} - \frac{a_1}{a_2} x_1\right) \right\}$

$$= \left\{ \left(x_1, -\frac{a_1}{a_2} x_1\right) + \left(0, \frac{b}{a_2}\right) \right\}$$

$$= \left\{ c\left(1,\ -\frac{a_1}{a_2}\right) + \underbrace{\left(0, \frac{b}{a_2}\right)}_{\text{a particular soln}} \right\}$$

Recall 1st order linear eqn

Homog: $y' + p(t)y = 0$

Seperable $\overset{\text{solve}}{\Longrightarrow}$ $y = Ce^{-\int p(t)dt}$

Nonhomog: $y' + p(t)y = g(t)$

$\overset{\Longrightarrow}{\text{int factor}}$ $y(t) = \dfrac{C}{I(t)} + \left(\underbrace{\dfrac{1}{I(t)}\int_{t_0}^{t} I(s)g(s)ds}_{\text{a particular soln}}\right)$

§ 2.1

$I(t) = e^{\int p(t)dt}$

Method of undetermined coefficients

Ex: $y'' - 6y' + 9y = e^t$

Gen soln: $y = \underbrace{C_1 y_1 + C_2 y_2}_{\text{Gen soln to } y''-6y'+9y=0} + y_p(t)$

char eqn: $r^2 - 6r + 9 = 0$

$(r-3)^2 = 0$

$r = 3, 3$

$y = C_1 e^{3t} + C_2 t e^{3t} + y_p(t)$

To find $y_p(t)$:

Ansatz: $y_p(t) = Ae^t$

Determine A: $y_p'' - 6y_p' + 9y_p = e^t$

$$(Ae^t)'' - 6(Ae^t)' + 9(Ae^t) = e^t$$

$$Ae^t - 6Ae^t + 9Ae^t = e^t$$

$$4Ae^t = e^t$$

$$4A = 1$$

$$A = \frac{1}{4}$$

$$y_p(t) = \frac{1}{4}e^t$$

$$\boxed{y = c_1 e^{3t} + c_2 t e^{3t} + \frac{1}{4}e^t}$$

End of lecture 1

How to make Ansatz?

Ex1: $y'' - 2y' + y' = t^3 e^{2t} \cos(4t)$

Char eq: $r^2 - 2r + 1 = 0$

$$(r-1)^2 = 0$$

$$r = 1, 1$$

$$y = c_1 e^t + c_2 t e^t + y_p(t)$$

Ansatz: $y_p(t) = (At^3 + Bt^2 + Ct + D)e^{2t}\cos(4t)$
$\qquad + (Et^3 + Ft^2 + Gt + H)e^{2t}\sin(4t)$

Determine A, B, C, D, E, F, G, H from

$$y_p'' - 2y_p' + y_p = t^3 e^{2t}\cos(4t)$$

Ex 2: $y'' - 3y' + 2y = t^2 e^{2t}$

char eq: $(r-1)(r-2) = 0$

$r = 1, 2$

coincides w/ $t^2 e^{2t}$

$$y = c_1 e^t + c_2 e^{2t} + y_p(t)$$

Ansatz: $y_p(t) = t(At^2 + Bt + C)e^{2t}$

Exercise Plug the incorrect Ansatz

$y_p = (At^2 + Bt + C)e^{2t}$ into the eqn,

see what happens

Ex3 $y'' - 6y' + 9y = e^{3t}$

char eq: $r^2 - 6r + 9 = 0$

$(r-3)^2 = 0$

$r = 3, 3$

coincide with e^{3t}

$$y = c_1 e^{3t} + c_2 t e^{3t} + y_p(t)$$

Ansatz: $y_p(t) = t^2 A e^{3t}$

Ex4 $y'' - 8y' + 25y = e^{4t}\cos(3t)$

char eq: $r^2 - 8r + 25 = 0$

$r = 4 \pm 3i$ each pair counts only once

$$y = c_1 e^{4t}\cos(3t) + c_2 e^{4t}\sin(3t) + y_p(t)$$

Ansatz: $y_p(t) = t\left(Ae^{4t}\cos(3t) + Be^{4t}\sin(3t)\right)$

Superposition

Ex: $y'' - y' = 2e^t + \cos t + t + te^t + 2\sin t$

$$y'' - y' = (2+t)e^t + (\cos t + 2\sin t) + t$$

char eq: $r^2 - r = 0$

$r(r-1) = 0$

$r = 0, 1$

$$y = C_1 + C_2 e^t + y_p(t)$$

Find particular soln's $y_{p_1}, y_{p_2}, y_{p_3}$ s.t.

Ansatz

$y_{p_1}'' - y_{p_1}' = (2+t)e^t$ $\qquad$ $y_{p_1} = t(At+B)e^t$

$y_{p_2}'' - y_{p_2}' = \cos t + 2\sin t$ $\qquad$ $y_{p_2} = C\cos t + D\sin t$

$y_{p_3}'' - y_{p_3}' = t$ $\qquad$ $y_{p_3} = t(Et+F)$

$$y_p(t) = y_{p_1} + y_{p_2} + y_{p_3}$$