Homogeneous Linear 2nd Order diff eqn's w/ Constant coefficients §3.1–3.4

Ex: $m\frac{dv}{dt} = -mg - \gamma v$ (1st order linear eqn)

$$m\frac{d^2y}{dt^2} = -mg - \gamma\frac{dy}{dt}$$

(y = position)

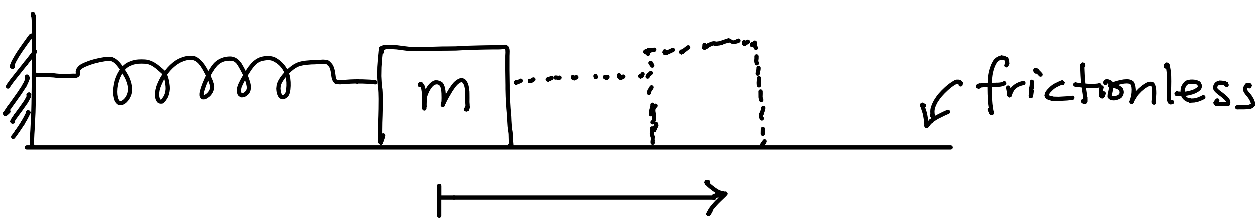
$$my'' + \gamma y' = -mg$$ (2nd order linear eqn)

const. coeff.

$\neq 0$, so not homog.

$y(t) = 0$ not a soln.

Ex (spring-mass system)



$u = 0$
equilibrium
position

u = displacement from equilibrium (positive when stretched)

$$m\frac{d^2u}{dt^2} = F_{spring} = -ku$$ (Hooke's law)

important comment said verbally in lecture but didn't write down: force due to spring always opposes displacement, so there's always an "–" sign regardless of our choice of coordinates

k = spring constant , $[k] = kg/s^2$

$$\boxed{mu'' + ku = 0}$$ (2nd order linear eqn)

const. coeff.   homog.

Indeed, $u(t) = 0$ is a soln.

Harmonic oscillator

Initial condition : $u(0)$ and $u'(0)$

Ex0 : $y' = 3y$, $y' - 3y = 0$

$y = Ae^{3t}$

Ex1 : $\boxed{y'' - 8y' + 15y = 0}$

Try to see if there are soln's of the form $y = e^{rt}$.

$$0 = r^2 e^{rt} - 8re^{rt} + 15e^{rt}$$

$$0 = e^{rt}(r^2 - 8r + 15)$$

charcteristic eqn: $\boxed{r^2 - 8r + 15 = 0}$

$$(r-3)(r-5)=0$$

Two distinct real roots : $r=3,\ 5$

$y_1 = e^{3t}$, $y_2 = e^{5t}$ are soln's.

$$(Ay_1+By_2)'' - 8(Ay_1+By_2)' + 15(Ay_1+By_2)$$

$$\overset{\text{linear}}{=} A\underbrace{(y_1''-8y_1'+15y_1)}_{=0} + B\underbrace{(y_2''-8y_2'+15y_2)}_{=0}$$

$$\overset{\text{homog.}}{=} 0$$

$\Rightarrow$ $Ay_1 + By_2$ are all solutions

(in fact, these are all the solutions)

$$L[y] = y'' - 8y' + 15y$$ linear operator

$$L[Ay_1+By_2] = A\,L[y_1] + B\,L[y_2]$$

$$\boxed{y = Ae^{3t} + Be^{5t}}$$

Arbitrary constants are determined by initial conditions $\boxed{y(0)=1\ ,\ \ y'(0)=2}$

$$y'(t) = 3Ae^{3t} + 5Be^{5t}$$

$$\left.\begin{aligned} 1 &= y(0) = A + B \\ 2 &= y'(0) = 3A + 5B \end{aligned}\right\} \quad \begin{aligned} &B = 1 - A \\ &2 = 3A + 5(1-A) \\ &2A = 3 \\ &A = \tfrac{3}{2}, \quad B = -\tfrac{1}{2} \end{aligned}$$

$$\boxed{y(t) = \frac{3}{2}e^{3t} - \frac{1}{2}e^{5t}}$$

End of lecture 1

Ex2 $\quad y'' + 4y' + 13y = 0$

char eqn: $r^2 + 4r + 13 = 0$

$$r = \frac{-4 \pm \sqrt{16-52}}{2} = \frac{-4 \pm \sqrt{-36}}{2}$$

$r = -2 \pm 3i$ (complex roots)

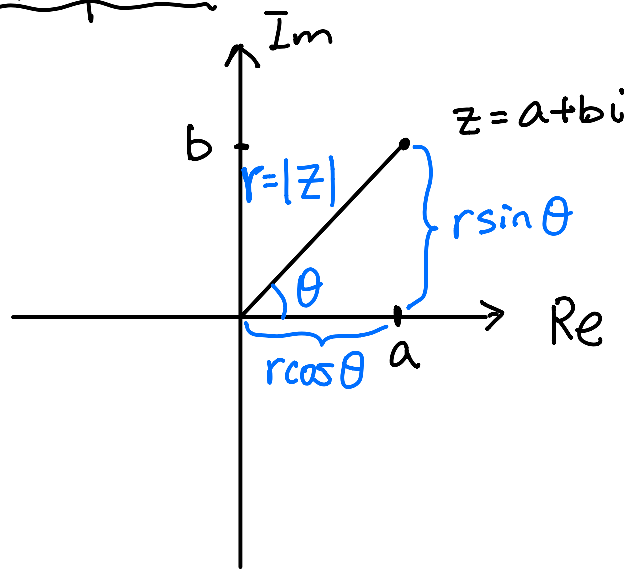
$$\boxed{y(t) = C_1 e^{(-2+3i)t} + C_2 e^{(-2-3i)t}}$$

Complex numbers and functions

$$z = a + ib$$

$$i^2 = -1$$

Complex plane



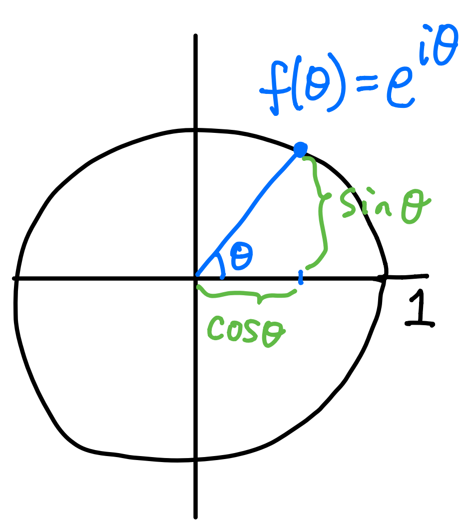
$$|z|^2 = |a+ib|^2 = a^2+b^2$$

$$|a+ib|^2 = (a+ib)(a-ib)$$
$$= a^2 + iba - iab + b^2 = a^2+b^2$$

$$\frac{1}{a+ib} = \frac{a-ib}{(a+ib)(a-ib)} = \frac{a-ib}{a^2+b^2}$$
$$= \frac{a}{a^2+b^2} - i\frac{b}{a^2+b^2}$$

Polar coordinates

$$z = a+ib$$
$$= r\cos\theta + ir\sin\theta$$



$$f(\theta) = \cos\theta + i\sin\theta$$

$$f'(\theta) = -\sin\theta + i\cos\theta$$

$$= i(\underbrace{\cos\theta + i\sin\theta}_{f(\theta)})$$

$f(\theta)$ satisfies $\left.\begin{matrix} f' = if \\ f(0) = 1 \end{matrix}\right\} \Rightarrow f(\theta) = e^{i\theta}$

Euler's formula

$$e^{i\theta} = \cos\theta + i\sin\theta$$

$$e^{\alpha+i\beta} = e^{\alpha}e^{i\beta} = e^{\alpha}(\cos\beta + i\sin\beta)$$

Another explanation for Euler's formula (optional)

$f(z) = e^z$ is defined by $\left\{\begin{matrix} f' = f \\ f(0) = 1 \end{matrix}\right\} \Rightarrow$ Taylor expansion of f at 0.

$$f(z) = e^z = 1 + z + \frac{z^2}{2!} + \frac{z^3}{3!} + \frac{z^4}{4!} + \cdots$$

$$f(i\theta) = e^{i\theta} = 1 + i\theta + \frac{(i\theta)^2}{2!} + \frac{(i\theta)^3}{3!} + \frac{(i\theta)^4}{4!} + \cdots$$

$$= 1 + i\theta - \frac{\theta^2}{2!} - i\frac{\theta^3}{3!} + \frac{\theta^4}{4!} + \cdots$$

$$= \left(1 - \frac{\theta^2}{2!} + \frac{\theta^4}{4!} + \cdots\right) + i\left(\theta - \frac{\theta^3}{3!} + \cdots\right)$$

$$e^{i\theta} = \cos\theta + i\sin\theta$$

$$e^{-i\theta} = \cos\theta - i\sin\theta$$

$$\cos\theta = \frac{e^{i\theta} + e^{-i\theta}}{2}$$

$$\sin\theta = \frac{e^{i\theta} - e^{-i\theta}}{2i}$$

Ex2 cont'd

$$\boxed{y(t) = C_1 e^{(-2+3i)t} + C_2 e^{(-2-3i)t}}$$

$$= C_1 e^{-2t}\left(\cos 3t + i\sin 3t\right) + C_2 e^{-2t}\left(\cos(-3t) + i\sin(-3t)\right)$$

$$= C_1 e^{-2t}\left(\cos 3t + i\sin 3t\right) + C_2 e^{-2t}\left(\cos 3t - i\sin 3t\right)$$

$$= \left(C_1 + C_2\right) e^{-2t}\cos 3t + \left(iC_1 - iC_2\right) e^{-2t}\sin 3t$$

$$\boxed{y(t) = A e^{-2t}\cos 3t + B e^{-2t}\sin 3t}$$

$$\begin{Bmatrix} A = C_1 + C_2 \\ B = iC_1 - iC_2 \end{Bmatrix} \iff \begin{Bmatrix} C_1 = \frac{A - iB}{2} \\ C_2 = \frac{A + iB}{2} \end{Bmatrix}$$

$$e^{-2t}\cos 3t = \frac{1}{2}\left(e^{(-2+3i)t} + e^{(-2-3i)t}\right)$$

$$e^{-2t}\sin 3t = \frac{1}{2i}\left(e^{(-2+3i)t} - e^{(-2-3i)t}\right)$$

End of Lecture 2

Ex 3: $y'' - 6y' + 9y = 0$

Char eqn: $r^2 - 6r + 9 = 0$

$$(r-3)^2 = 0$$

$$r = 3, 3$$

e^{3t} is a soln, but we need another

Solve diff. eqn iteratively

Differential operator D is defined as

$$Dy = y'$$

$$D^2y = D(Dy) = Dy' = y''$$

Ex 3 cont'd

$$0 = y'' - 6y' + 9y$$

$$= D^2y - 6Dy + 9y$$

$$= (D^2 - 6D + 9)y$$

check! $\rightarrow$ $= (D-3)(D-3)\,y$ $= (D-3)(y'-3y)$

$= D(y'-3y) - 3(y'-3y)$

$= y'' - 3y' - 3y' + 9y$

$= y'' - 6y' + 9y$

$$(D-3)\underbrace{(D-3)y}_{=u} = 0$$

$$\left[\begin{aligned} &(D-3)u = 0 \\ &u' - 3u = 0 \\ &u' = 3u \\ &u = Be^{3t} \end{aligned}\right.$$

$$u = (D-3)y$$

$$Be^{3t} = y' - 3y$$

$$y' - 3y = Be^{3t}$$ (1st order linear eqn)
see Ex 1 in §2.1 lecture

$$\boxed{y = Ae^{3t} + Bte^{3t}}$$

Nonhomog. eqn (§3.5)

Ex: $y'' - 6y' + 9y = e^t$

$$(D^2-6D+9)y = e^t$$

$$(D-3)\underbrace{(D-3)y}_{=u} = e^t$$

$$(D-3)u = e^t$$

$$u' - 3u = e^t \quad \text{(1st order linear eqn)}$$

Exercise: Solve for u

$$u = -\tfrac{1}{2}e^t + Be^{3t}$$

$$-\tfrac{1}{2}e^t + Be^{3t} = u = (D-3)y$$

$$y' - 3y = -\tfrac{1}{2}e^t + Be^{3t} \quad \text{(1st order linear eqn)}$$

Exercise: Solve for y

$$\boxed{y(t) = Ae^{3t} + Bte^{3t} + \tfrac{1}{4}e^t}$$