Autonomous eqn §2.5

$$\frac{dy}{dt} = f(y)$$

Ex: $\frac{dP}{dt} = rP$ (autonomous)

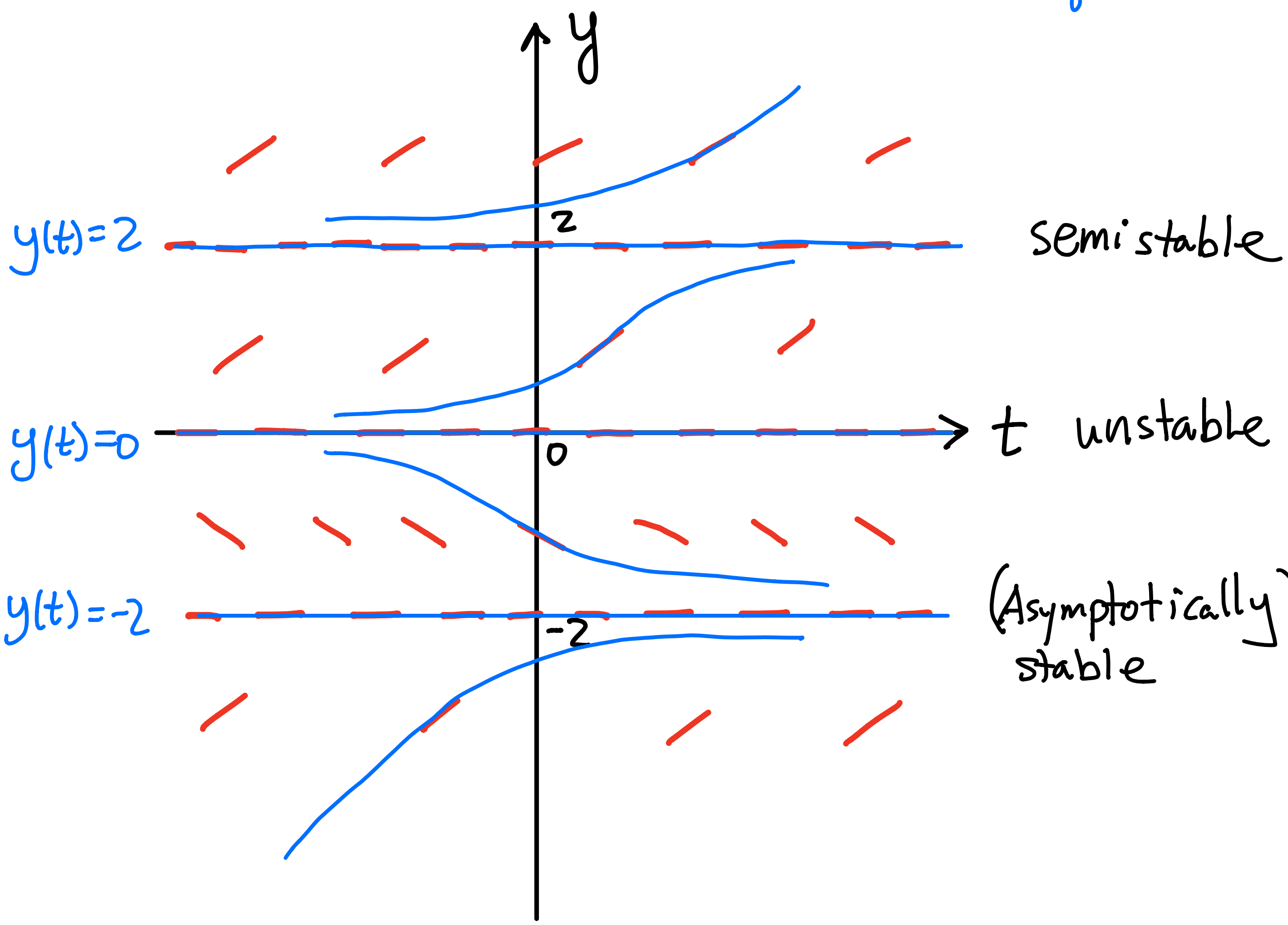
$\frac{dy}{dt} = 3y + 7e^{3t}$ (not autonomous)

Direction field

Ex: $\frac{dy}{dt} = 8(y^2-4)^2(y^2+2y)$

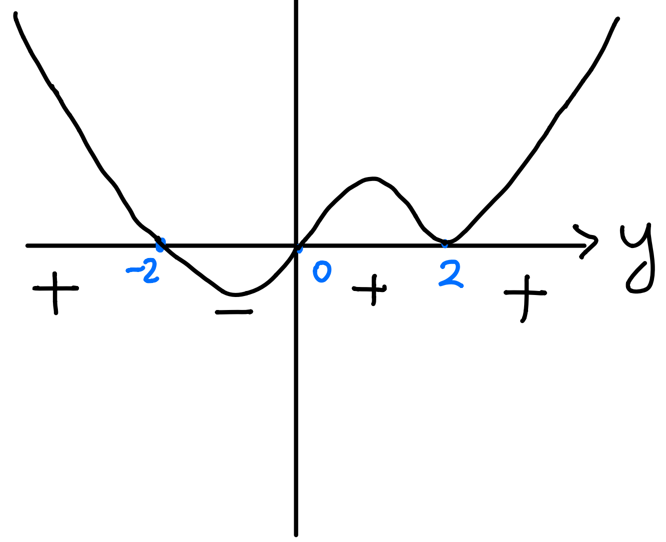
$= 8(y+2)^3(y-2)^2 y$

$\frac{dy}{dt} = 0$ when $y = -2, 0, 2$ $\leftrightarrow$ constant soln's (i.e. equilibrium soln's)



To figure out the sign of $\frac{dy}{dt}$, can graph the Polyonomial

$8(y+2)^3(y-2)^2 y = 8y^6 + \cdots$

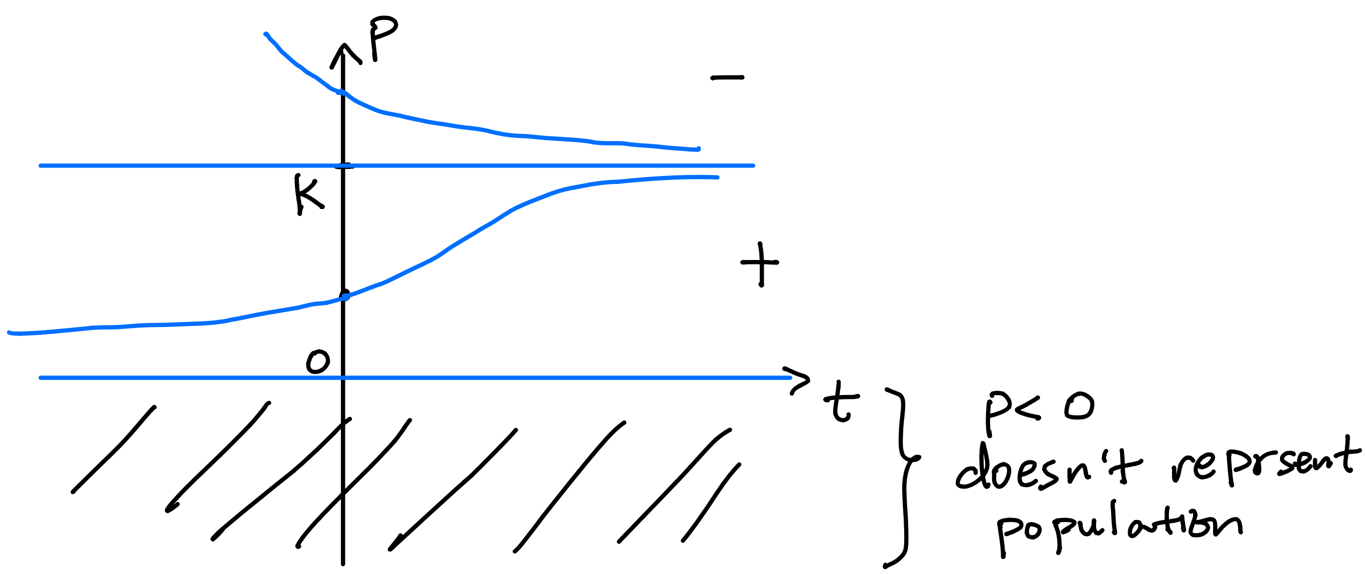


Population growth

Exponential growth : $\frac{dP}{dt} = rP$

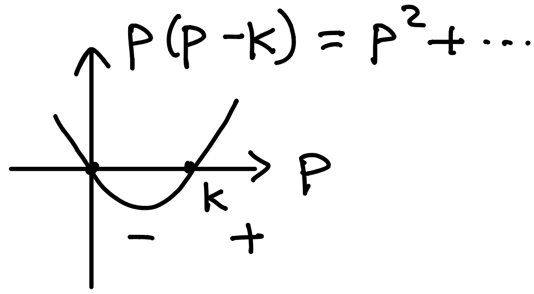
In an ecosystem, when crowded, growth reaches a limit K = environmental carrying capacity.

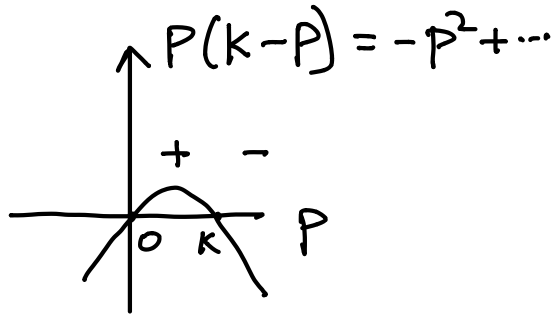
$[K] = [P]$



$$\frac{dP}{dt} = \frac{r}{K}(K-P)P$$

$$\boxed{\frac{dP}{dt} = r\left(1-\frac{P}{K}\right)P}$$ Logistic growth





Autonomous $\Rightarrow$ separable

$$\int \frac{dP}{\left(1-\frac{P}{K}\right)P} = \int r\,dt$$

Partial fraction

$$\frac{1}{P\left(1-\frac{P}{K}\right)} = \frac{A}{P} + \frac{B}{1-\frac{P}{K}} = \frac{A\left(1-\frac{P}{K}\right)+BP}{P\left(1-\frac{P}{K}\right)}$$

$$A = 1 \quad , \quad -\frac{AP}{K} + BP = 0$$
$$\Rightarrow B = \frac{1}{K}$$

$$\int\left(\frac{1}{P} + \frac{1/K}{1-P/K}\right)dP = \int r\,dt$$

$$\int \frac{\frac{1}{K}}{1-\frac{P}{K}}\,dP = -\int \frac{1}{u}\,du$$

$$u = 1-\frac{P}{K}$$
$$du = -\frac{1}{K}\,dP$$

$$\ln|P| - \ln\left|1-\frac{P}{K}\right| = rt + C$$

$$\ln\left|\frac{P}{1-\frac{P}{K}}\right| = rt + C$$

$$\frac{P}{1-\frac{P}{k}} = De^{rt}$$

$$P = De^{rt}\left(1-\frac{P}{k}\right)$$

$$P + \frac{P}{k}De^{rt} = De^{rt}$$

$$P\left(1+\frac{1}{k}De^{rt}\right) = De^{rt}$$

$$P = \frac{De^{rt}}{1+\frac{1}{k}De^{rt}}$$

$$\boxed{P(t) = \frac{k}{Ee^{-rt}+1}} \qquad E = \frac{k}{D}$$

Comment added after the lecture: you are allowed to use these formulas we found for p(t) and E in hw and exams

$$\boxed{P_0 = P(0)} = \frac{k}{E+1} \quad \Rightarrow \quad \boxed{E = \frac{k-P_0}{P_0}}$$

As $t \to \infty$, $p(t) \to k$

Variants

① Simplistic model for epidemics

$y(t)$ = # of infected individuals (in percent)

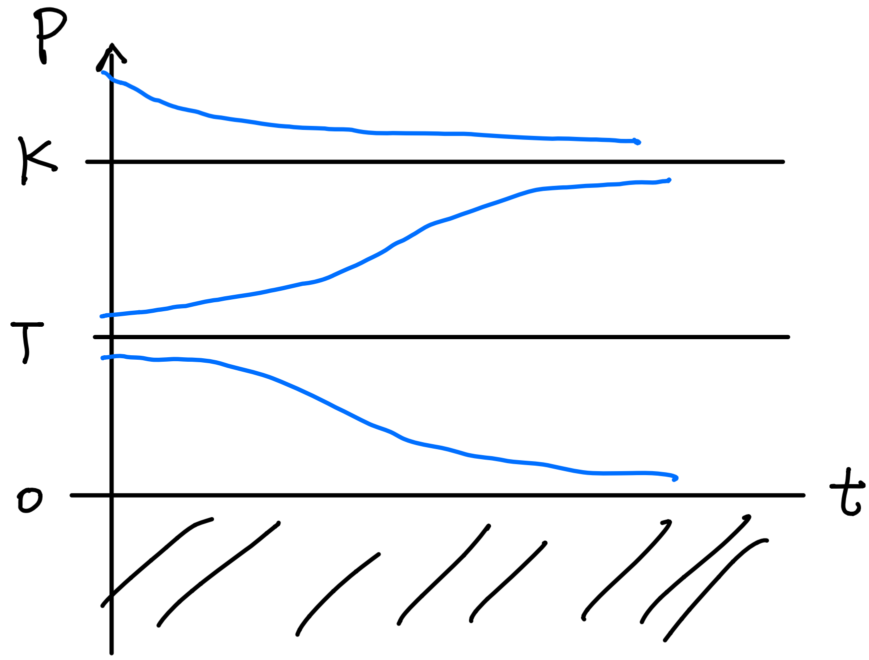
$1-y$ = # of noninfected individuals

$$\frac{dy}{dt} = \alpha y(1-y)$$

② Sustainable harvesting

$$\frac{dy}{dt} = \underbrace{r\left(1-\frac{y}{K}\right)y}_{\text{logistic growth}} \underbrace{- Ey}_{\text{harvesting}}$$

③ Population model with threshold



P < 0 don't care

<u>Exercise</u> : $\frac{dP}{dt} = -r\left(1-\frac{P}{T}\right)\left(1-\frac{P}{K}\right)P$