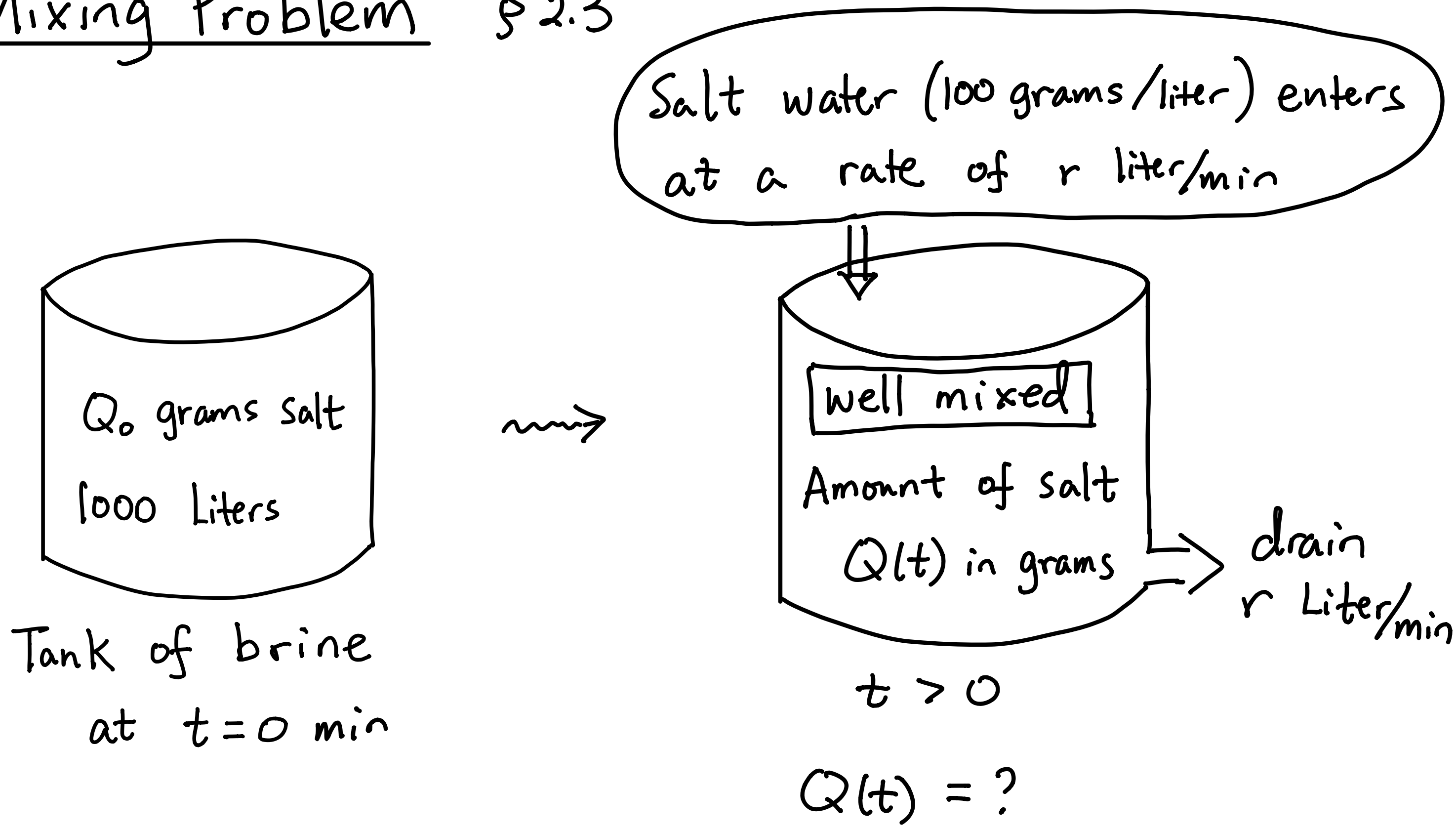
Mixing Problem §2.3



(a) Write down an IVP for Q(t).

$$\boxed{Q(0) = Q_0} \text{ grams}$$

$$\frac{dQ}{dt} = \text{rate in} - \text{rate out}$$

$$\text{rate in} = \left(100 \frac{\text{gram}}{\cancel{\text{liter}}}\right)\left(r \frac{\cancel{\text{liter}}}{\text{min}}\right) = 100r \text{ gram/min}$$

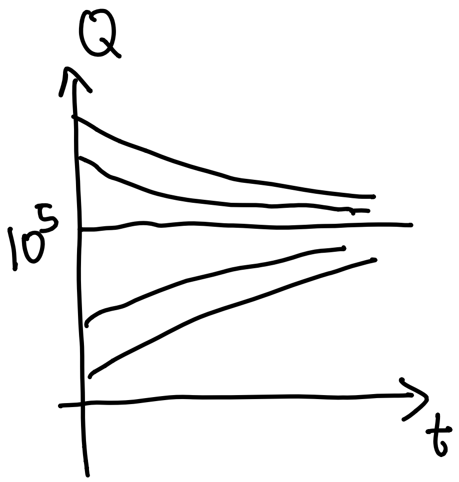
$$\text{rate out} = \left(\frac{Q(t)\text{ gram}}{1000\ \cancel{\text{liter}}}\right)\left(r \frac{\cancel{\text{liter}}}{\text{min}}\right) = \frac{Q(t)\, r}{1000} \text{ gram/min}$$

$$\boxed{\frac{dQ}{dt} = 100r - \frac{Qr}{1000}}$$

(b) Discuss the solution to the above IVP.

Separable : exercise to solve

$$Q(t) = 10^5 + (Q_0 - 10^5) e^{-\frac{rt}{1000}}$$



* As $t \to \infty$, $Q(t) \to 10^5$ grams

* $Q(t) = 10^5$ g is an equilibrium soln

$Q' = 0$ when $100r - \frac{Qr}{1000} = 0$

i.e. $Q = 10^5$

* $Q(t) \to 10^5$ more rapidly as r increases.

(c) Let $q(t)$ = the amount of salt in the tank in kg.

Write down an IVP for $q(t)$

$q(t)$ kg $\qquad Q(t)$ gram

when $q = 1$, $Q = 1000$

$$Q = 1000q$$

$$\frac{dQ}{dt} = 100r - \frac{Qr}{1000}$$

$$\frac{d(1000q)}{dt} = 100r - \frac{(1000q)r}{1000}$$

$$1000\frac{dq}{dt} = 100r - qr$$

$$\boxed{\frac{dq}{dt} = 0.1r - \frac{qr}{1000}} \quad \frac{kg}{min}$$

$$q(0) = Q_0 \text{ ~~gram~~} \frac{1\,kg}{1000 \text{ ~~gram~~}} = \frac{Q_0}{1000}\,kg$$

$$\boxed{q(0) = \frac{Q_0}{1000}}$$

(d) Write down an IVP for the concentration of salt $C(t)$ (in gram/liter) in the tank.

$c(t)\ \frac{gram}{liter}$, $Q(t)$ grams in 1000 liters

$$c(t) = \frac{Q(t)}{1000}\ \frac{grams}{liters}$$

$$Q = 1000c$$

$$\frac{dQ}{dt} = 100r - \frac{Qr}{1000}$$

$$\frac{d(1000c)}{dt} = 100r - \frac{(1000c)r}{1000}$$

$$\boxed{\frac{dc}{dt} = 0.1r - \frac{cr}{1000}} \quad \frac{gram/liter}{min}$$

$$\boxed{c(0) = \frac{Q_0}{1000}\ \frac{gram}{liter}}$$

(e) Write down an IVP for the concentration of salt $\mu(t)$ (in kg/liter) in the tank.

$\mu(t)$ $\frac{\text{kg}}{\text{liter}}$ $\qquad$ $C(t)$ $\frac{\text{grams}}{\text{liter}}$

when $\mu = 1$, $C = 1000$

$$C = 1000\,\mu$$

$$\frac{dC}{dt} = 0.1r - \frac{Cr}{1000}$$

$$\frac{d(1000\mu)}{dt} = 0.1r - \frac{(1000\,\mu)r}{1000}$$

$$1000\frac{d\mu}{dt} = 0.1r - \mu r$$

$$\boxed{\frac{d\mu}{dt} = \frac{0.1r}{1000} - \frac{\mu r}{1000}} \qquad \frac{\text{kg/liter}}{\text{min}}$$

$$\mu(0) = \frac{C(0)}{1000}$$

$$\boxed{\mu(0) = \frac{Q_0}{10^6}} \qquad \frac{\text{kg}}{\text{liter}}$$

(f) Write down an IVP for the concentration $C(t)$ in g/liter but with time, s, measured in hours.

s hr $\qquad$ t min

when $s = 1$ $\quad t = 60$

$$t = 60s \quad \Rightarrow s = \frac{t}{60} \quad \Rightarrow \frac{ds}{dt} = \frac{1}{60}$$

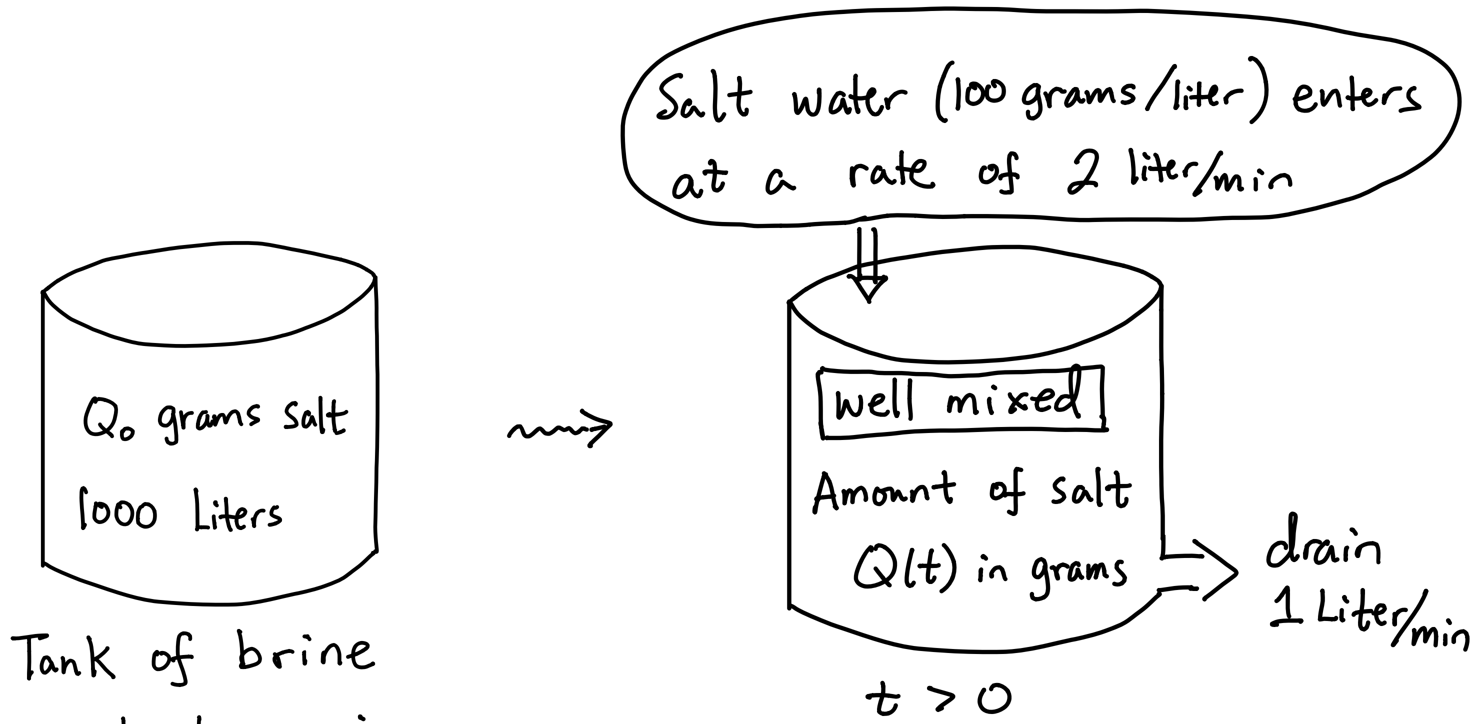
$$\frac{1}{60}\frac{dc}{ds} = \frac{dc}{ds}\frac{ds}{dt} = \frac{dc}{dt} = 0.1r - \frac{cr}{1000}$$

$$\frac{dc}{ds} = 60\left(0.1r - \frac{cr}{1000}\right)$$

$$\boxed{\frac{dc}{ds} = 6r - \frac{3cr}{50}} \quad \frac{\text{gram/liter}}{\text{hr}}$$

$$\boxed{C(0) = \frac{Q_0}{1000}\ \frac{\text{gram}}{\text{liter}}}$$

Time dependent mixing problem



$Q(t) = ?$

$$\frac{dQ}{dt} = \text{rate in} - \text{rate out}$$

$$\text{rate in} = \left(100\ \frac{\text{gram}}{\text{liter}}\right)\left(2\ \frac{\text{liter}}{\text{min}}\right) = 200\ \text{gram}/\text{min}$$

$$\text{rate out} = \left(\frac{Q(t)\ \text{gram}}{1000+t}\right)\left(1\ \frac{\text{liter}}{\text{min}}\right) = \frac{Q}{1000+t}\ \text{gram}/\text{min}$$

(until the tank fills up)

$$\boxed{\frac{dQ}{dt} = 200 - \frac{Q}{1000+t}}$$