§2.1 First order linear equations cont'd

Method of integrating factors

For 1st order linear equations:

$$p(t)\frac{dy}{dt} + Q(t)\,y = G(t), \quad y(t_0) = y_0$$

$$\frac{dy}{dt} + \frac{Q(t)}{P(t)}\,y = \frac{G(t)}{P(t)}$$

$$y' + q(t)\,y = g(t)$$

$$I(t)\left(y' + q(t)\,y\right) = I(t)\,g(t)$$

Find $I(t)$ s.t.

$$\cancel{Iy'} + Iqy = I(t)\,(y' + qy) = \frac{d\left(I(t)\,y\right)}{dt} = \cancel{Iy'} + I'y$$

$$\frac{dI}{dt} = q(t)\,I$$

$$\int \frac{dI}{I} = \int q(t)\,dt$$

$$\ln|I| = \int q(t)\,dt$$

$$\boxed{I(t) = e^{\int q(t)\,dt}}$$

Can pick $I(t) = e^{\int_{t_0}^{t} q(s)\,ds}$, so $I(t_0) = 1$

$$\frac{d(I(t)\,y)}{dt} = I(t)\,g(t)$$

Comment added after the lecture, (also mentioned briefly last time):

Note that if we name $u = I(t)\,y$ as mentioned in the last lecture.

Then $\frac{du}{dt} = I(t)\,g(t)$.

So u satisfies a simple eqn of the form $\frac{du}{dt} = f(t)$.

The soln is $u = \int f(t)\,dt$.

This is the point of this method.

$$\int d(I(t)y) = \int I(t)\,g(t)\,dt$$

$$I(t)\,y = \int I(t)\,g(t)\,dt$$

$$\boxed{y(t) = \frac{1}{I(t)} \int I(t)\,g(t)\,dt}$$

$$y(t) = \frac{1}{I(t)}\left(\int_{t_0}^{t} I(s)\,g(s)\,ds + C\right)$$

Exercise: Soln to $y' + q(t)y = 0$

$$\boxed{y(t) = \frac{1}{I(t)} \int_{t_0}^{t} I(s)\,g(s)\,ds + \frac{C}{I(t)}}$$ (General soln)

$$y_0 = y(t_0) = \frac{1}{\underbrace{I(t_0)}_{=1}}(0 + C) = C$$

$$\boxed{y(t) = \frac{1}{I(t)}\left(\int_{t_0}^{t} I(s)\,g(s)\,ds + y_0\right)}$$ (Soln to IVP)

Conclusion : Existence & Uniqueness theorem for 1st order linear eqn

Suppose $q(t)$, $g(t)$ are continuous on an interval $(a, b) \ni t_0$,

$\Rightarrow$ then for any choice of initial value $y(t_0) = y_0$, there exists a unique solution $y(t)$ on (a, b) satisfying

$$y' + q(t)\, y = g(t), \quad y(t_0) = y_0$$

Ex : $ty' + 2y = 4t^2, \quad y(1) = a, \quad t > 0$

$$y' + \frac{2}{t} y = 4t$$

$$I(t) = e^{\int q(t)\,dt} = e^{\int \frac{2}{t}\,dt} = e^{2\ln|t|} = |t|^2 = t^2$$

$$t^2 y' + 2ty = 4t^3$$

$$\frac{d(t^2 y)}{dt} = 4t^3$$

$$\int d(t^2 y) = \int 4t^3\, dt$$

$$t^2 y = t^4 + C$$

$$y(t) = t^2 + \frac{C}{t^2}$$

Or directly using formula:

$$y(t) = \frac{1}{I(t)} \int I(t)\, g(t)\, dt$$

$$= \frac{1}{t^2} \int t^2 4t \, dt = \frac{1}{t^2} \int 4t^3 dt$$

$$= \frac{1}{t^2} (t^4 + C)$$

$$= t^2 + \frac{C}{t^2}$$

$$a = y(1) = 1 + \frac{C}{1} \Rightarrow C = a - 1$$

$$\boxed{y(t) = t^2 + \frac{a-1}{t^2}}$$

As $t \to \infty$, $y(t) \to \infty$

As $t \to 0$, $y(t) \to \infty$, $a - 1 > 0$, i.e. $a > 1$

$y(t) \to -\infty$, $a - 1 < 0$, i.e. $a < 1$

$y(t) \to 0$, $a - 1 = 0$, i.e. $a = 1$

<u>Remark</u>: $ty' + 2y = 4t^2$

when $t=0$, $\quad 0 + 2y(0) = 0$

$\Rightarrow y(0) = 0$

<u>Direction field</u>: $y' = 4t - \frac{2y}{t}$

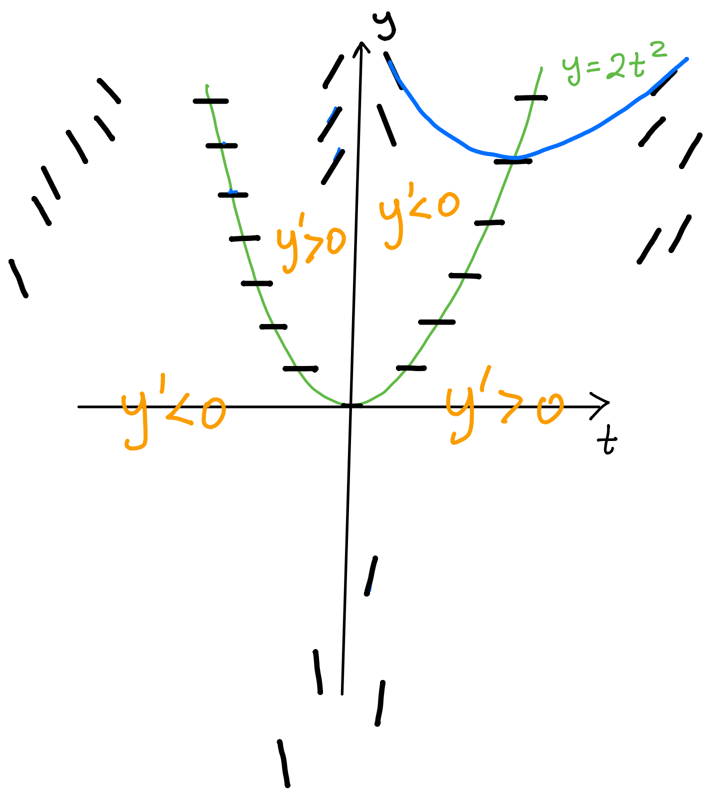
* $y'=0$, when $4t - \frac{2y}{t} = 0$, i.e. $\underbrace{y = 2t^2}_{\uparrow}$

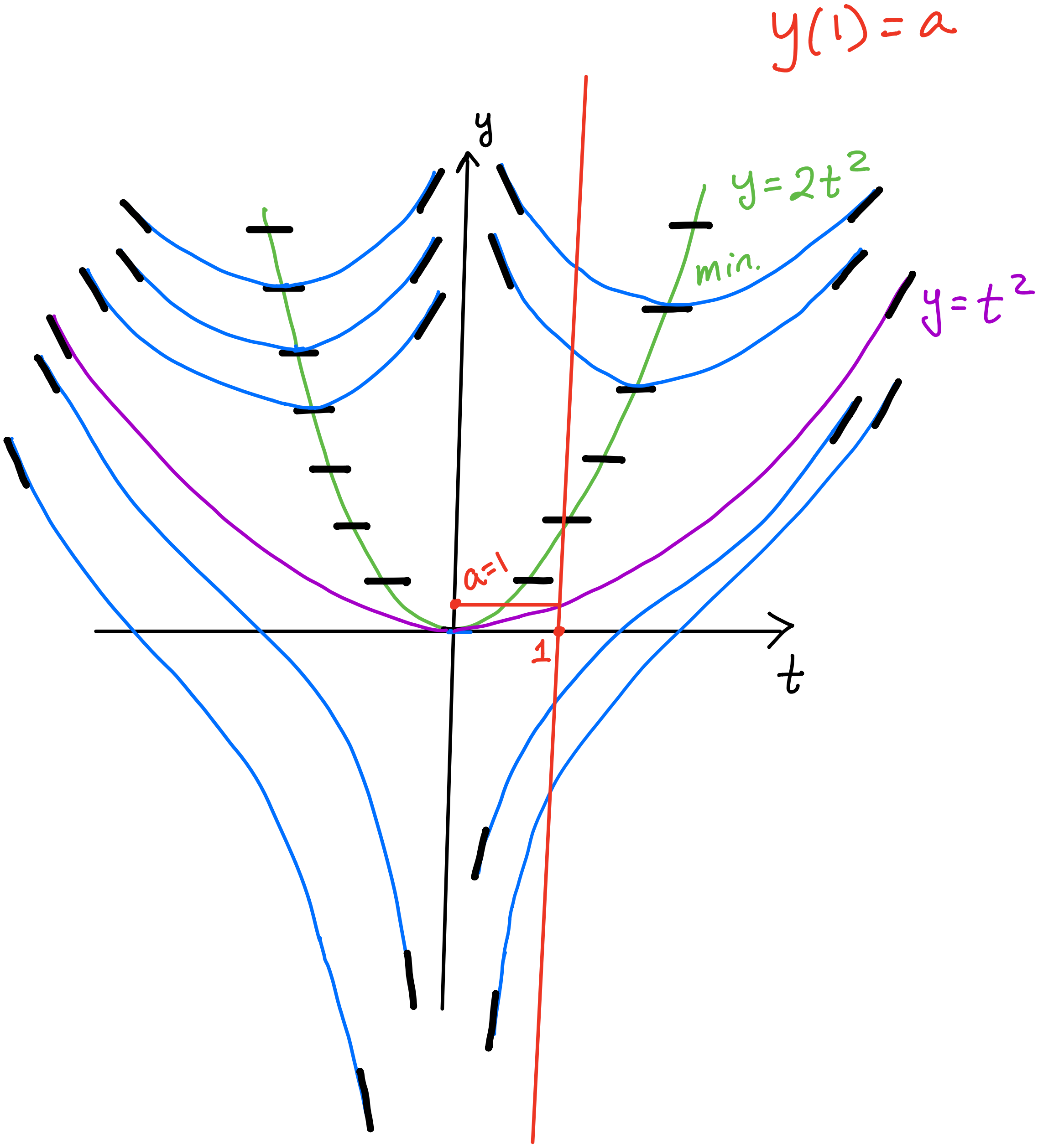
not a soln curve

* $y'>0$, when $4t - \frac{2y}{t} > 0$

$\Rightarrow \frac{y}{t} < 2t \Rightarrow \begin{cases} y < 2t^2 & \text{when } t>0 \\ y > 2t^2 & \text{when } t<0 \end{cases}$

* $y'<0$, otherwise





<u>Ex</u>: IVP with nonlinear eqn may not have a unique soln

$$\frac{dy}{dt} = y^{1/3}, \quad y(0) = 0, \quad t \geq 0$$

separable: $\int y^{-1/3}\,dy = \int dt$

$$\frac{3}{2}y^{2/3} = t + C$$

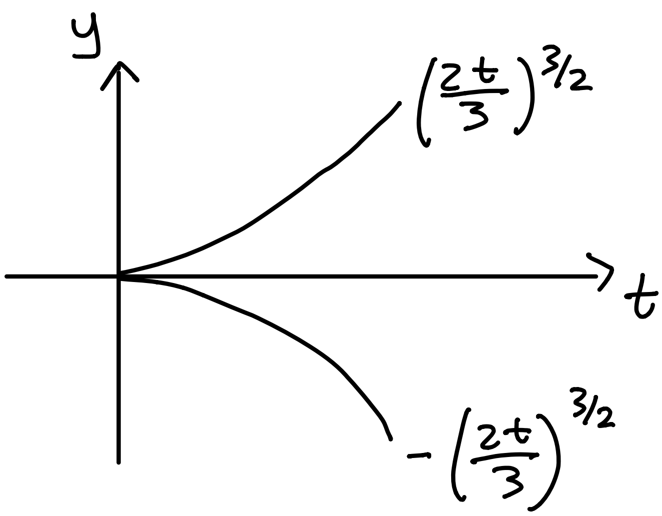
$$y^2 = \left[\frac{2}{3}(t+c)\right]^3$$

$$y = \pm\left[\frac{2}{3}(t+c)\right]^{3/2}$$

$$y(0)=0 \quad \Rightarrow \quad 0 = y(0) = \pm\left[\frac{2}{3}c\right]^{3/2}$$

$$\Rightarrow \quad C = 0$$

$$y = \pm\left(\frac{2t}{3}\right)^{3/2}$$



in fact, there are even more soln's see Ex3 in §2.4

Comment after the lecture: forgot to mention that it can be immediately seen the constant function $y(t)=0$ is also a solution. (there are many more soln in Ex 3 §2.4)