§2.1 Linear first order ODE

linear eqn

* $f(x,y) = ax + by - c = 0$

  i.e. $y = \frac{c}{b} - \frac{a}{b}x$ (a line in $\mathbb{R}^2$)

* $f(x,y,z) = ax + by + cz - d = 0$

  (a plane in $\mathbb{R}^3$)

* $f(x,y,z,u,v) = ax + by + cz + du + ev - g = 0$

Nonlinear eqn ex:

* $f(x,y) = x^2 - y = 0$, i.e. $y = x^2$ not linear
* $f(x,y) = xy - 1 = 0$, i.e. $y = \frac{1}{x}$ not linear

Eqn linear in the dependent variables

$$f(\overbrace{x, y, z, u, v}^{\text{dept}}, \overset{\text{indept}}{t})$$

$$= a(t)x + b(t)y + c(t)z + d(t)u + e(t)v - g(t) = 0$$

linear in x, y, z, u, v

Linear ODE

$$f(\overbrace{y, y', y'', \dots, y^{(n)}}^{\text{dept.}}, \underset{\text{indept.}}{t})$$

$$= a_n(t)\, y^{(n)}(t) + \cdots + a_1(t)\, y'(t) + a_0(t)\, y(t) - g(t) = 0$$

linear in $y, y', \dots, y^{(n)}$

Ex

$P' = rP$ (linear)

$P' = rP - 450$ (linear)

$v' = -10 - \frac{v}{5}$ (linear)

$v' = -10 + \frac{v^2}{5}$ (nonlinear)

$y''(t) + \sin(t+y) = \sin t$ (nonlinear)

$y'' + (\sin t) y = \sin t$ (linear)

$t y''' + y' y = 0$ (nonlinear)

Solving Linear 1st order eqn

Ex 0 : $\frac{dy}{dt} - 3y = 0$, $y(0) = 9$

linear and separable.

$$\int \frac{dy}{y} = \int 3\, dt$$

$$\ln|y| = 3t + C$$

$$\boxed{y(t) = Be^{3t}}$$ (general soln)

$$9 = y(0) = B$$

$$\boxed{y(t) = 9e^{3t}}$$ (soln to IVP)

Ex 1 $\frac{dy}{dt} - 3y = 7e^{3t}$, $y(0) = 9$

Not separable, but linear

$$I(t)(y' - 3y) = I(t)\, 7e^{3t}$$

Find $I(t)$ s.t.

$$\cancel{I}y' - 3Iy = I(t)(y' - 3y) = \frac{d(I(t)\,y)}{dt} = \cancel{I}y' + I'y$$

$$I' = -3I$$

$$I(t) = e^{-3t}$$ (just pick any soln)

$$\frac{d(I(t)\,y)}{dt} = I(t)\,7e^{3t}$$

$u = I(t)\,y$

$$\frac{du}{dt} = I(t)\,7e^{3t}$$

$$u = \int I(t)\,7e^{3t}\,dt$$

$$\int d(I(t)\,y) = \int I(t)\,7e^{3t}\,dt$$

$$u = I(t)\,y = \int I(t)\,7e^{3t}\,dt$$

$$y = \frac{1}{I(t)}\int I(t)\,7e^{3t}\,dt = \frac{1}{e^{-3t}}\int e^{-3t}\,7e^{3t}\,dt$$

$$= e^{3t}\left(\int 7\,dt\right) = e^{3t}(7t + B)$$

$$\boxed{y(t) = 7te^{3t} + Be^{3t}}$$ (general solution)

$= e^{3t}(7t+B) \to \infty$ as $t \to \infty$ for all B.

$9 = y(0) = B \Rightarrow \boxed{y(t) = 7te^{3t} + 9e^{3t}}$ (soln to IVP)

Remark

$y' - 3y = 0$ (homogeneous eqn)

($y = 0$ is a soln)

$y' - 3y = 7e^{3t} \neq 0$ (nonhomogeneous eqn)

($y = 0$ is not a soln)

Homog $y' - 3y = 0$

If $y(t)$ satisfies $y' - 3y = 0$,

then $B\,y(t)$ satisfies

$$(By)' - 3(By) = By' - 3By = B(\underbrace{y' - 3y}_{=0}) = 0$$

So $B\,y(t)$ is also a soln.

Nonhomog $y' - 3y = 7e^{3t}$

If $y(t)$ satisfies $y' - 3y = 7e^{3t}$,

then $B\,y(t)$ satisfies

$$(By)' - 3(By) = By' - 3By = B(\underbrace{y' - 3y}_{7e^{3t}}) = B7e^{3t}$$

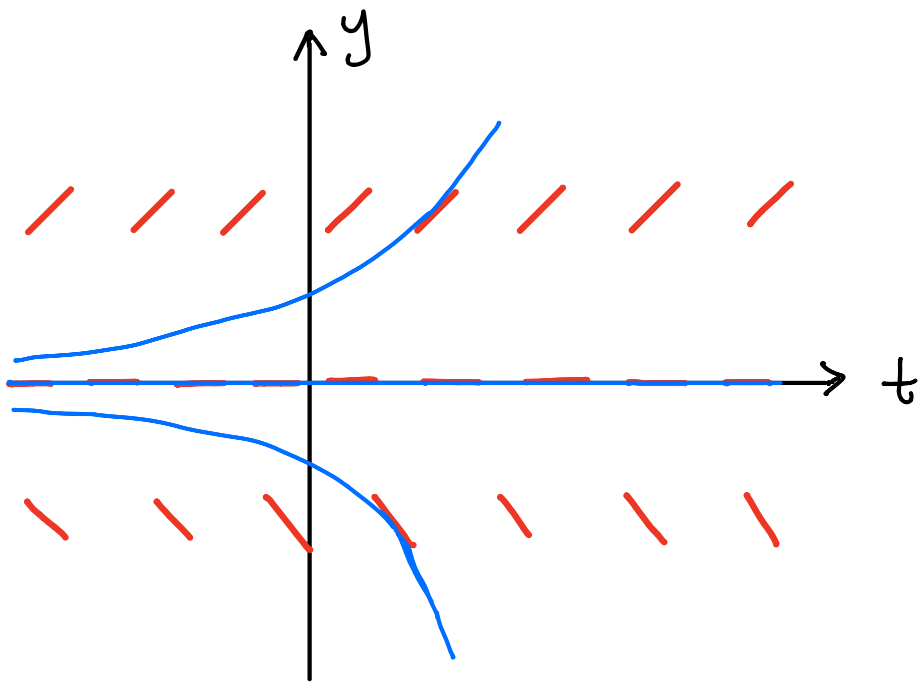
So $B\,y(t)$ is not a soln.

$$(7te^{3t} + Be^{3t})' - 3(7te^{3t} + Be^{3t}) = 7e^{3t}$$

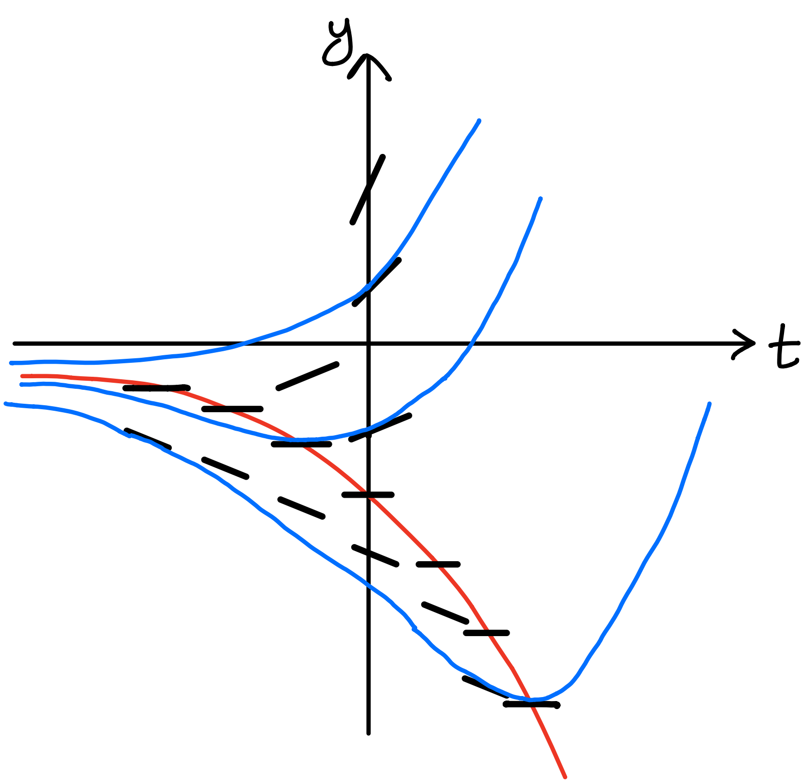
$$\underbrace{(7te^{3t})' - 3(7te^{3t})} + [\underbrace{(Be^{3t})' - 3(Be^{3t})}_{=0}] = \underline{7e^{3t}}$$

Direction field

Ex 0 : $y' = 3y$



Ex 1 $y' = 3y + 7e^{3t} = 3\left(y + \frac{7}{3}e^{3t}\right)$



* $y' = 0$ when $y = -\frac{7}{3}e^{3t}$

Note: $y = -\frac{7}{3}e^{3t}$ not a solution curve.

* $y(t) \to \infty$ as $t \to \infty$ for all soln's