Newton's law of motion and laws for forces

Ex vertical motion experiencing gravity and air resistance

$y(t) = $ position $\left(\begin{array}{l}\text{choose where } y=0 \\ \text{choose direction } + = \text{up}\end{array}\right)$

$v(t) = \frac{dy}{dt}$ velocity

$a(t) = \frac{dv}{dt} = \frac{d^2y}{dt^2}$ acceleration

$mv = $ momentum (measure of "inertia")

mass

Newton's law of motion

$$F := \frac{d}{dt}(mv) = m\frac{dv}{dt}$$

assume mass is constant

Force

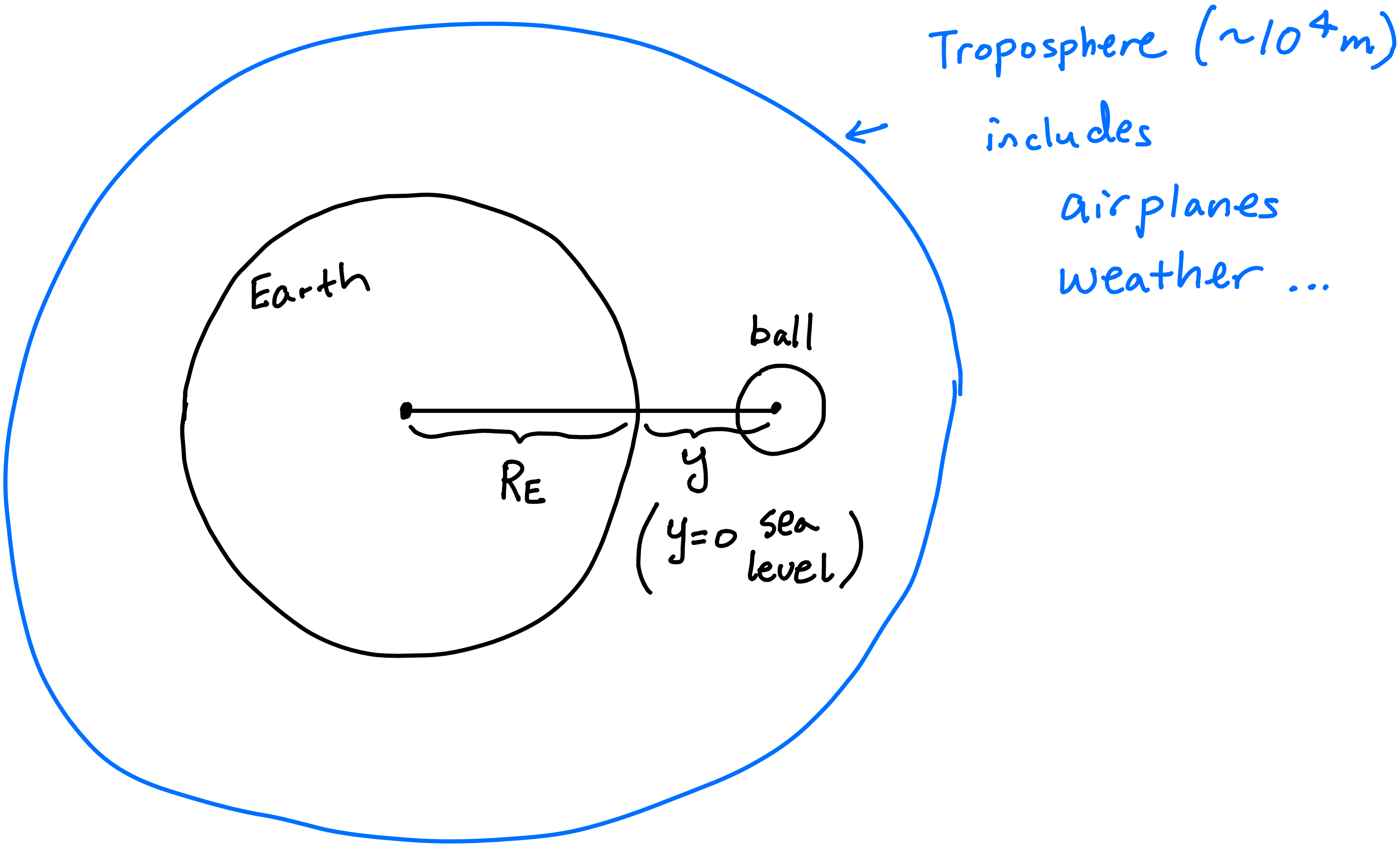
depends on laws for forces in various settings

$$F = ma \quad \text{or} \quad F = m\frac{dv}{dt} \quad \text{or} \quad F = m\frac{d^2y}{dt^2}$$

Gravity only (ignore air resistance for now)

$m\frac{dv}{dt} = -mg$, i.e. $\cancel{m}\frac{dv}{dt} = -9.8\cancel{m}$, $v(t) = -9.8t + C$

$g = $ acceleration due to gravity $\approx 9.8\ m/s^2$



$$F_{gravity}(y) = -\frac{k}{(R_E + y)^2}$$

$$-mg = F_{gravity}(0) = -\frac{k}{R_E^2}$$

$$\Rightarrow k = mg R_E^2$$

$$F_{gravity}(y) = -\frac{mg R_E^2}{(R_E + y)^2} \approx -mg$$

$$\left.\begin{aligned} R_E &\sim 10^6 \text{ m} \\ y &\lesssim 10^4 \text{ m} \end{aligned}\right\} \Rightarrow R_E + y \approx R_E$$

Include air resistance (More generally drag force)

$$m\frac{dv}{dt} = -mg + F_{drag}$$

shape, size, ambient fluid viscosity, material, etc

$$F_{drag} = \alpha(v, \dots)\, v^2$$

$=$

- Slow (nonairborne dust, ball in honey)

  $= -\gamma v(t)$ $\gamma = \text{constant} > 0$

- fast (ball in air, water)

  $$= \left\{ \begin{array}{ll} -\gamma v^2 & \left(\begin{array}{l}\text{obj. moving}\\ \text{upwards}\end{array}\right) \\ +\gamma v^2 & \left(\begin{array}{l}\text{obj. moving}\\ \text{downwards}\end{array}\right) \end{array} \right\} = -\gamma v|v|$$

- other behaviors

(Slow) $m\frac{dv}{dt} = -mg - \gamma v(t)$

$$\frac{dv}{dt} = -10 - \frac{v}{5}$$

Exercise 1 For $\frac{dv}{dt} = -10 - \frac{v}{5}$

(a) plot the direction field in a $v(t)$ vs. t graph.

(b) Draw a few solution curves with different $v(0)$

(c) Please verbally describe each of the solutions in (b)

(d) Describe the behavior of $v(t)$ as $t \to \infty$

(e) For $v(0) = 10$, when does the object reach the top?

Exercise 2 For an object thrown upwards modeled by $\frac{dv}{dt} = -10 - \frac{v^2}{5}$,

(a) once it reaches the top, it will start falling downwards, what differential equation should we use to model the downward part of the motion?

(b) As it falls, what is its terminal velocity?

End of Week 2's lecture 1.

Solution to these exercises will be discussed in the next lecture.

Start of the first part of lecture 2, soln to exercises

Exercise 1 For $\frac{dv}{dt} = -10 - \frac{v}{5}$

(a) plot the direction field in a $v(t)$ vs. t graph.

(b) Draw a few solution curves with different $v(0)$

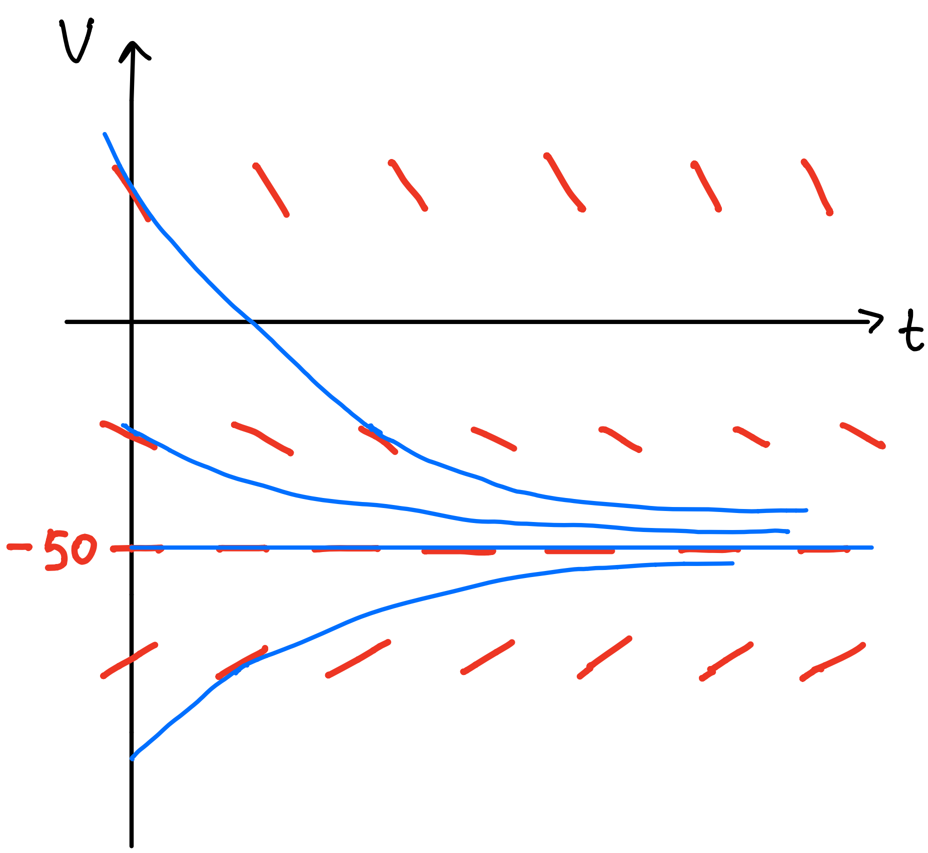
(c) Please verbally describe each of the solutions in (b)

(a, b, c) $\frac{dv}{dt} = 0$ when $-10 - \frac{v}{5} = 0$, i.e. $v = -50$

Verbal part see video

$\frac{dv}{dt} > 0$ when $v < -50$

$\frac{dv}{dt} < 0$ when $v > -50$



(d) Describe the behavior of $v(t)$ as $t \to \infty$

$v(t) \to -50$ as $t \to \infty$ for all solutions

(e) For $v(0) = 10$, when does the object reach the top?

$$\frac{dv}{dt} = -\frac{1}{5}(50+v)$$

$$\int \frac{dv}{v+50} = \int -\frac{1}{5}\,dt$$

$$\ln|v+50| = -\frac{1}{5}t + C$$

$$v+50 = A e^{-t/5}$$

$$v = -50 + A e^{-t/5}$$

$$10 = v(0) = -50 + A \quad \Rightarrow \quad A = 60$$

$$\boxed{v(t) = -50 + 60 e^{-t/5}}$$

The object reaches the top when $v=0$.

$$0 = v(t) = -50 + 60 e^{-t/5}$$

$$e^{-t/5} = \frac{50}{60}$$

$$-\frac{t}{5} = \ln\left(\frac{5}{6}\right)$$

$$t = 5\ln\left(\frac{6}{5}\right) \approx \boxed{0.91}$$

Exercise 2 For an object thrown upwards modeled by $\frac{dv}{dt} = -10 - \frac{v^2}{5}$,

(a) once it reaches the top, it will start falling downwards, what differential equation should we use to model the downward part of the motion?

$$\frac{dv}{dt} = -10 + \frac{v^2}{5}$$

(b) As it falls, what is its terminal velocity?

When $\frac{dv}{dt} = -10 + \frac{v^2}{5} = 0$

$$\Rightarrow \quad v^2 = 50$$

$$\Rightarrow \quad v = -5\sqrt{2}$$

Terminology

Ordinary diff. eqn (ODE)

has only 1 indept. variable

Ex of partial differential eqn (PDE)

$$\frac{\partial u(x,t)}{\partial t} = \alpha^2 \frac{\partial^2 u(x,t)}{\partial x^2} \quad \text{(heat diffusion)}$$