Mirror Symmetry for Theta Divisors

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Caltech/USC Algebra & Geometry Seminar

[ACLL]

Haniya Azam, Catherine Cannizzo, Heather Lee, Chiu-Chu Melissa Liu

Complex Geometry

theta divisors Θ in principally polarized abelian varieties

$\longleftarrow$ Mirror Symmetry $\longrightarrow$

Symplectic Geometry

Landau-Ginzburg model (Y, W)

$W : Y \to \mathbb{C}$

Complex moduli of theta divisors in principally polarized abelian varieties (ppav's)

Siegel upper half space $\mathcal{H}_g := \{ \tau = B + i\Omega \in S_g(\mathbb{C}) \mid \Omega \text{ positive definite} \}$

$\mathcal{H}_g$: moduli of ppav's of $\dim_{\mathbb{C}} = g$ with Torelli structure

$S_g(\mathbb{C})$: $g \times g$ symmetric matrices

Abelian variety V_τ of complex dimension g

$$V_\tau = (\mathbb{C}^*)^g / \tau\mathbb{Z}^g \ni (x_1, \dots, x_g) = (e^{2\pi i z_1}, \dots, e^{2\pi i z_g})$$

$$\updownarrow \qquad \qquad \uparrow \exp$$

$$V_\tau^+ = \mathbb{C}^g / \mathbb{Z}^g + \tau\mathbb{Z}^g \ni (z_1, \dots, z_g)$$

multiplicative action: $\tau n \cdot (x_1, \dots, x_g) = (e^{2\pi i (\tau n)_1} x_1, \dots, e^{2\pi i (\tau n)_g} x_g)$

Theta divisor Θ_τ

line bundle $\mathcal{L}_\tau = (\mathbb{C}^*)^g \times \mathbb{C} / \tau\mathbb{Z}^g \longrightarrow V_\tau = (\mathbb{C}^*)^g / \tau\mathbb{Z}^g$

$$\tau n \cdot (x_1, \dots, x_g, v) = \left(e^{2\pi i (\tau n)_1} x_1, \dots, e^{2\pi i (\tau n)_g} x_g,\ e^{-\pi i n^T \tau n} x_1^{-n_1} \cdots x_g^{-n_g} v \right)$$

Riemann theta function $\vartheta(\tau, \cdot) : (\mathbb{C}^*)^g \longrightarrow \mathbb{C}$

$$\vartheta(\tau, x) = \sum_{n \in \mathbb{Z}^g} x_1^{n_1} \cdots x_g^{n_g} e^{\pi i n^T \tau n}$$

descends to a section $\vartheta \in H^0(V_\tau, \mathcal{L}_\tau) = \mathbb{C}\vartheta$

$\Theta_\tau := \{ \vartheta(\tau, x) = 0 \} \subseteq V_\tau = (\mathbb{C}^*)^g / \tau\mathbb{Z}^g$ (when $g=2$ Θ_τ genus 2 curve)

Cannizzo's thesis:

HMS when $g=2$, $\tau = \frac{i}{2\pi} \log t \begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix}$

$t > 0$ large

Principal polarization

$$c_1(\mathcal{L}_\tau) = [\omega_{V_\tau}] \in H^{1,1}(V_\tau) \cap H^2(V_\tau, \mathbb{Z})$$

An integral symplectic basis $\{\alpha_j, \beta_j\}_{j=1}^{g}$ of $\left(\underset{\substack{\mathbb{Z}^{2g} \\ \shortparallel}}{H_1(V_\tau, \mathbb{Z})}, [\omega_{V_\tau}]\right)$

with dual basis $\{a_j, b_j\}$ s.t. $[\omega_{V_\tau}] = \sum_{j=1}^{g} a_j \cup b_j$

$\alpha_j, \beta_j : [0,1] \to (\mathbb{C}^*)^g$

$\alpha_j(s) = (1, \dots, 1, \underbrace{e^{2\pi i s}}_{j\text{ th}}, 1, \dots, 1)$, $\beta_j(s) = \left(e^{2\pi i \tau_{j1} s}, \dots, e^{2\pi i \tau_{jg} s}\right)$

General polarization: $[\omega_{(\delta_1, \dots, \delta_g)}] = \sum_1 \delta_j a_j \cup b_j$, $\delta_j \in \mathbb{Z}$, $\delta_j \mid \delta_{j+1}$

Moduli $\mathcal{A}_g$ of g dimensional ppav's $(V_\tau, c_1(\mathcal{L}_\tau))$

Moduli of ppav

+ Torelli structure

choice of integral symplectic basis $\{\alpha_j, \beta_j\}$ of

$$\left(H_1(V_\tau, \mathbb{Z}), c_1(\mathcal{L}_\tau)\right) \cong \left(\mathbb{Z}^{2g}, J = \begin{bmatrix} 0 & I \\ -I & 0 \end{bmatrix}\right)$$

$$\mathcal{H}_g \longrightarrow \mathcal{A}_g = [\mathcal{H}_g / Sp(2g, \mathbb{Z})]$$

$\circlearrowleft$ $Sp(2g, \mathbb{Z})$

$$\begin{bmatrix} A & C \\ E & D \end{bmatrix} \circ \tau = (A\tau + C)(E\tau + D)^{-1}$$

Pure tropical Siegel space

$$\mathbb{H}_g = \underbrace{S_g(\mathbb{R})}_{\mathbb{R}^{g(g+1)/2}} \times \mathbb{H}_g^{trop,p} := \{\Omega \in S_g(\mathbb{R}) \mid \Omega \text{ positive definite}\}$$

[Chan – Melo – Viviani 2013]

$$[\mathbb{H}_g / S_g(\mathbb{Z})] = [\underbrace{S_g(\mathbb{R})/S_g(\mathbb{Z})}_{(S^1)^{g(g+1)/2}} \times \mathbb{H}_g^{trop,p}]$$

$$\begin{tikzcd}
\mathbb{H}_g \arrow[d] \arrow[dr] & \\
{[\mathbb{H}_g / S_g(\mathbb{Z})]} \arrow[r] \arrow[d] & \mathbb{H}_g^{trop,p} \arrow[d] \\
A_g^F := [\mathbb{H}_g / P_g(\mathbb{Z})] \arrow[r] \arrow[d] & A_g^{trop,p} = [\mathbb{H}_g^{trop,p} / GL_g(\mathbb{Z})] \\
A_g = [\mathbb{H}_g / Sp(2g;\mathbb{Z})] &
\end{tikzcd}$$

moduli of pure tropical ppav's

$GL_g(\mathbb{Z}) \ni h$ action:

$h\Omega = h\Omega h^T$

Kähler moduli of Y

mirror LG model (Y, W)

Theorem [ACLL] Under homological mirror symmetry,

(1) $\mathcal{H}_g^{trop,P}$ is the kähler space $K(Y)$ of Y.

space of all kähler classes

$\left(S_g(\mathbb{R})/S_g(\mathbb{Z})\right) \times \mathcal{H}_g^{trop,P}$ is the complexified kähler space $K_{\mathbb{C}}(Y)$ of Y

(2) $A_g^{trop,P}$ is the kähler moduli of Y

$K(Y)/\mathrm{Aut}(Y)$

A_g^F is the complexified kähler moduli of Y.

(3) when $g=2$, $\dim K(Y) = 3$

kähler cones $\longleftrightarrow$ 3 cones in Voronoi decomposition

Siegel parabolic subgroup

$$P_g(\mathbb{Z}) = \left\{ \begin{bmatrix} A & C \\ 0 & D \end{bmatrix} \in Sp(2g,\mathbb{Z}) \right\} = \left\{ \begin{bmatrix} A & C \\ 0 & (A^T)^{-1} \end{bmatrix} : A \in GL_g(\mathbb{Z}),\ AC^T = CA^T \right\}.$$

* $P_g(\mathbb{Z})$ is generated by the following two subgroups

$$\left\{ \begin{bmatrix} I & C \\ 0 & I \end{bmatrix} : C^T = C \right\} \cong S_g(\mathbb{Z}) \qquad \tau \mapsto \tau + C$$

$$\left\{ \begin{bmatrix} A & 0 \\ 0 & (A^T)^{-1} \end{bmatrix} : A \in GL_g(\mathbb{Z}) \right\} \cong GL_g(\mathbb{Z}) \qquad \tau \mapsto A\tau A^T$$

* $Sp(2g,\mathbb{Z})$ is generated by $P_g(\mathbb{Z})$ and $\begin{pmatrix} 0 & I \\ -I & 0 \end{pmatrix}$ $\qquad \tau \mapsto -\tau^{-1}$

Ppav with a SYZ fibration (V_τ, π_τ^{SYZ})

$$\begin{tikzcd}
(x_1,\dots,x_g) \in (\mathbb{C}^*)^g \arrow[r] \arrow[d, mapsto] \arrow[d] & V_\tau = (\mathbb{C}^*)^g / \tau\mathbb{Z}^g \arrow[d, "\pi_\tau^{SYZ}"] & T_F \text{ fiber} \arrow[l] \\
(\xi_1,\dots,\xi_g) \in \mathbb{R}^g \arrow[r] & \mathbb{R}^g/\Omega\mathbb{Z}^g &
\end{tikzcd}$$

$\xi_j = \frac{1}{2\pi}\log|x_j|$

base of SYZ fibration

Lagrangian Sublattice $\Gamma_F = H_1(T_F,\mathbb{Z}) \subseteq H_1(V_\tau,\mathbb{Z}) \cong \mathbb{Z}^{2g}$

$\mathbb{Z}$-basis $\{\alpha^j\}_{j=1,\dots,g}$ $\qquad$ $\{\alpha_j, \beta_j\}_{j=1,\dots,g}$ $\mathbb{Z}$-sympletic basis

$[\omega_{V_\tau}](\alpha^j, \beta_k) = \delta^j_k$

$[\omega_{V_\tau}](\alpha^j, \alpha_k) = 0$

$P_g(\mathbb{Z})$ is the subgroup preserving Γ_F

$A_g^F := [\mathcal{H}_g / P_g(\mathbb{Z})]$ moduli of pairs (V_τ, Γ_F)

moduli of principally polarized and SYZ fibered abelian variety

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$$
\begin{tikzcd}
\mathcal{H}_g = S_g(\mathbb{R}) \times \mathcal{H}_g^{trop,p} \arrow[d] \arrow[dr] & \\
\left[\mathcal{H}_g / S_g(\mathbb{Z})\right] = \left[S_g(\mathbb{R})/S_g(\mathbb{Z}) \times \mathcal{H}_g^{trop,p} \right] \arrow[r] \arrow[d] & \mathcal{H}_g^{trop,p} \arrow[d] \\
A_g^F := \left[\mathcal{H}_g / P_g(\mathbb{Z})\right] \arrow[r, "\text{SYZ fibered ppav} \,\longmapsto\, \text{its base}"] \arrow[d] & A_g^{trop,p} = \left[\mathcal{H}_g^{trop,p} / GL_g(\mathbb{Z})\right] \\
A_g = \left[\mathcal{H}_g / Sp(2g;\mathbb{Z})\right] &
\end{tikzcd}
$$

Landau-Ginzburg mirror (Y_τ, W)

[Hori-Vafa 2000] ....

[Abouzaid-Auroux-Katzarkov 2020]

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$$\boxed{\text{LG model } (\tilde{Y}_\tau, \tilde{W}) \text{ mirror to } \tilde{\Theta}_\tau = \Big\{ \vartheta(\tau, x) = \sum_{n \in \mathbb{Z}^g}' x_1^{n_1} \cdots x_g^{n_g} e^{\pi i n^T \tau n} \Big\} \subseteq (\mathbb{C}^*)^g}$$

$$\Theta_\tau \subseteq V_\tau = (\mathbb{C}^*)^g / \tau \mathbb{Z}^g$$

SYZ mirror to $Bl_{\tilde{\Theta}_\tau \times \{0\}} (\mathbb{C}^*)^g \times \mathbb{C}$ $\quad |x_j| = e^{2\pi \xi_j}$

Moment polyhedron for $\tilde{Y}_\tau$

$$\Delta_\Omega = \Big\{ (\xi, \eta) \in \mathbb{R}^{g+1} \;\Big|\; \eta \geq \varphi(\xi) = \max_{n \in \mathbb{Z}^g} \{ \langle \xi, n \rangle + \underbrace{k(n)}_{} \} \Big\}$$

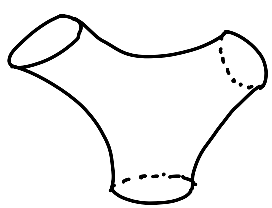
$\xi \in \mathbb{R}^g$ $\qquad k(n) = -\frac{1}{2} n^T \Omega n$

$$= \bigcap_{n \in \mathbb{Z}^g} \Big\{ (\xi, \eta) \in \mathbb{R}^{g+1} \;\Big|\; L_n(\xi, \eta) = -\langle \xi, n \rangle + \eta - k(n) \geq 0 \Big\}$$

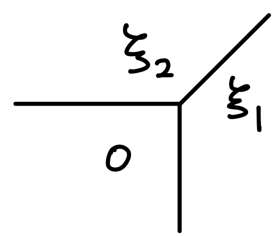
polyhedron with facets $\{ L_n(\xi, \eta) = 0 \}$ normal to $\nu_m = \begin{pmatrix} -n_1 \\ \vdots \\ -n_g \\ 1 \end{pmatrix}$

<u>Ex</u>:

$\{ 1 + x_1 + x_2 \} \subseteq (\mathbb{C}^*)^2$



$\eta \geq \max\{0, \xi_1, \xi_2\}$



moment polyhedron for $\mathbb{C}^3$

Complex structure on $\tilde{Y}_\tau$ (invariant under $(\mathbb{C}^*)^{g+1}$)

* $\mathbb{C}^{g+1}$ charts indexed by the vertices of Δ_Ω
* Facets of Δ_Ω corresponds to toric divisors $\{D_n\}_{n\in\mathbb{Z}^g}$
* $\tilde{W}: \tilde{Y}_\tau \to \mathbb{C}$ vanishes to order 1 on each toric divisors (= product of the coordinates in each chart)

Generic fiber: $\tilde{W}^{-1}(c) \cong (\mathbb{C}^*)^g$, $c \neq 0$

Singular fiber: $\tilde{W}^{-1}(0) = \bigcup_{n\in\mathbb{Z}^g} D_n$

* Each vertex of Δ_Ω is of the form $\sigma = \sigma_{n^{(1)},\dots,n^{(g)}} = \bigcap_{j=1}^{g+1} \{L_{n^{(j)}}(\xi,\eta) = 0\}$
* Transition maps are obtained from:

complex toric coordinates $t \in (\mathbb{C}^*)^{g+1} \leftrightsquigarrow$ inhomogeneous coordinates $y^\sigma \in \mathbb{C}^{g+1}$

$$t_1 = \prod_{j=1}^{g+1} (y_j^\sigma)^{-n_1^{(j)}}, \dots, \quad t_g = \prod_{j=1}^{g+1} (y_j^\sigma)^{-n_g^{(j)}}, \quad t_{g+1} = y_1^\sigma \cdots y_{g+1}^\sigma$$

* $\tilde{W}$ = holomorphic function extending t_{g+1}

Symplectic structure on $\tilde{Y}_\tau^\epsilon = \{\tilde{W}^{-1}(|z|<\epsilon)\}$ (invariant under $U(1)^{g+1}$)

[Guillemin 1994], [Kanazawa-Lau 2016]

Dual Kähler potential $\tilde{G} : \Delta_\Omega \to \mathbb{R}$

$$\tilde{G}(\xi,\eta) = \sum_{n\in\mathbb{Z}^g} \chi_n(\xi,\eta)\, L_n(\xi,\eta) \log L_n(\xi,\eta)$$

Kähler form: $\omega = \sum_{k=1}^{g+1} d\xi_k \wedge d\theta_k$ $\qquad \xi_{g+1} := \eta$

For $t \in (\mathbb{C}^*)^g$, $t_j = e^{2\pi(\rho_j + i\theta_j)}$, $\rho_j = \dfrac{\partial G}{\partial \xi_j}$

$\tilde{W} : Y_\tau^\epsilon \to \mathbb{C}$ is a symplectic fibration away from the singular fiber $\tilde{W}^{-1}(0)$

B-field (invariant under $U(1)^{g+1}$)

$B = \mathrm{Re}(\tau)$ determines $[B_\tau] \in H^2(\tilde{Y}_\tau;\mathbb{R})$ via injective map $i^* : H^2(\tilde{Y}_\tau;\mathbb{R}) \to H^2(\tilde{W}^{-1}(\varepsilon);\mathbb{R})$

$$i^* B_\tau = \sum_{j,k=1}^{g} B_{jk}\, dr_j \wedge d\theta_k$$

$\Omega r = \xi$

LG model (Y_τ, v_0) mirror to Θ_τ

$\tau\mathbb{Z}^g$ action on $\widetilde{Y}_\tau$: $(\tau n)\cdot(t_1,\dots,t_g,t_{g+1}) = (t_{g+1}^{-n_1}t_1,\dots,t_{g+1}^{-n_g}t_g, t_{g+1})$

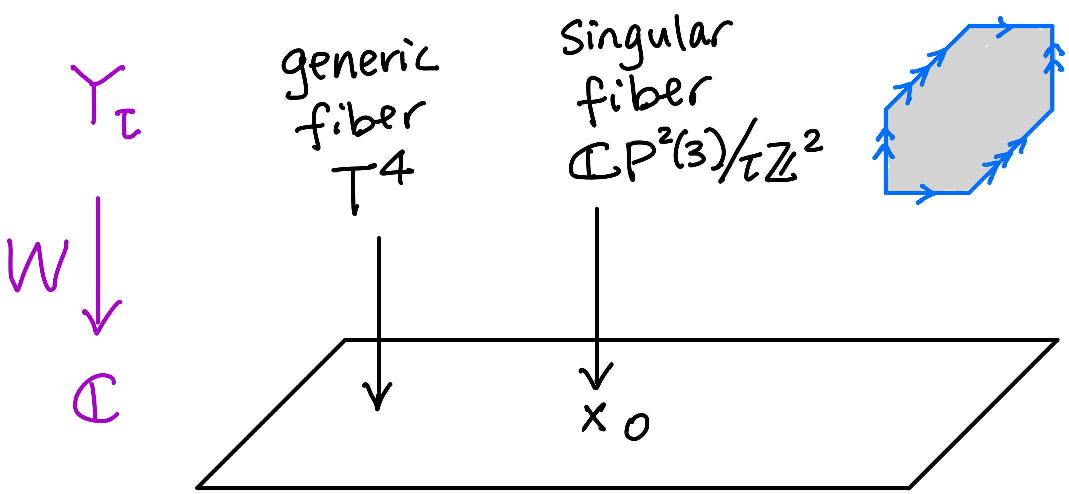
* preserves the complex structure and $\widetilde{W}$
* On $\widetilde{Y}_\tau^\epsilon$, it is free, and it preserves ω, B

$Y_\tau := \widetilde{Y}_\tau^\epsilon / \tau\mathbb{Z}^g$

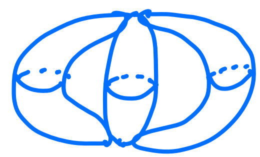
$W : Y_\tau \to \mathbb{C}$, generic fiber $\cong T^{2g}$

ω, B descends to Y

When $g=2$ $\quad \Theta_\tau$ = genus 2 curve



Critical locus



Chamber I

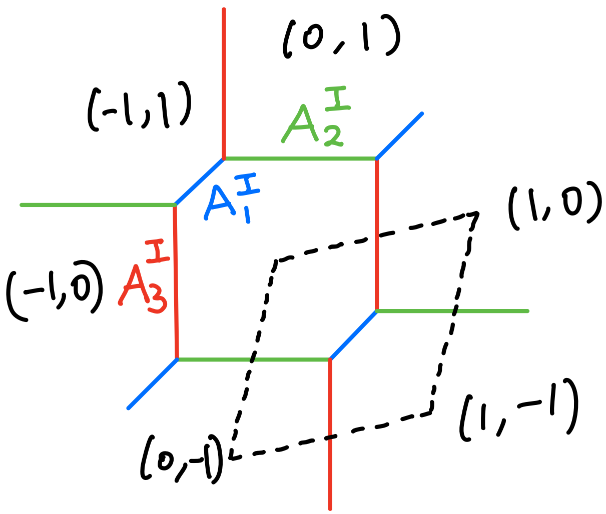
$$\begin{cases} \Omega_{12} = \Omega_{21} > 0 \\ \Omega_{11} - \Omega_{12} > 0 \\ \Omega_{22} - \Omega_{12} > 0 \end{cases}$$

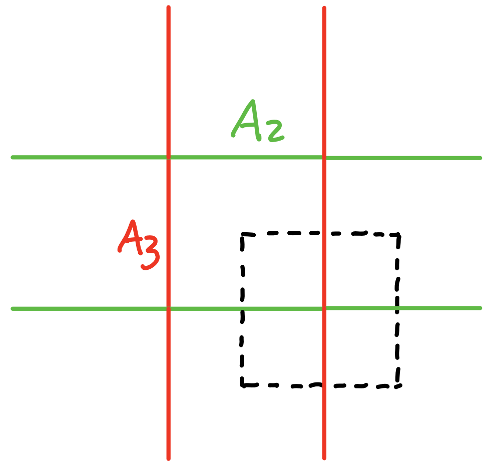
$$\begin{pmatrix} \Omega_{11} & \Omega_{12} \\ \Omega_{21} & \Omega_{22} \end{pmatrix} = \begin{pmatrix} A_1^{I} + A_2^{I} & A_1^{I} \\ A_1^{I} & A_1^{I} + A_3^{I} \end{pmatrix}$$

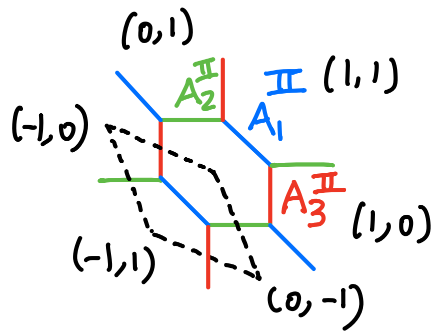
Chamber II

$$\begin{cases} \Omega_{12} = \Omega_{21} < 0 \\ \Omega_{11} - \Omega_{12} > 0 \\ \Omega_{22} - \Omega_{12} > 0 \end{cases}$$

$$\begin{pmatrix} \Omega_{11} & \Omega_{12} \\ \Omega_{21} & \Omega_{22} \end{pmatrix} = \begin{pmatrix} A_1^{II} + A_2^{II} & -A_1^{II} \\ -A_1^{II} & A_1^{II} + A_3^{II} \end{pmatrix}$$







Atiyah flop example

$$\Omega = \begin{pmatrix} 1+\lambda & \lambda \\ \lambda & 1+\lambda \end{pmatrix}$$

$\lambda > 0$

$A_1^{I} = \lambda$

$A_2^{I} = A_3^{I} = 1$

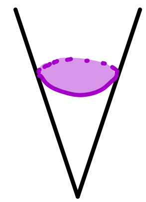
$\lambda < 0$

$A_1^{II} = -\lambda$

$A_2^{II} = A_3^{II} = 1 + 2\lambda$

$$\Omega = \begin{pmatrix} \Omega_{11} & \Omega_{12} \\ \Omega_{21} & \Omega_{22} \end{pmatrix} = \begin{pmatrix} z+y & x \\ x & z-y \end{pmatrix}$$

positive definite $\iff \{(x,y,z) \in \mathbb{R}^3 \mid z > \sqrt{x^2+y^2}\}$



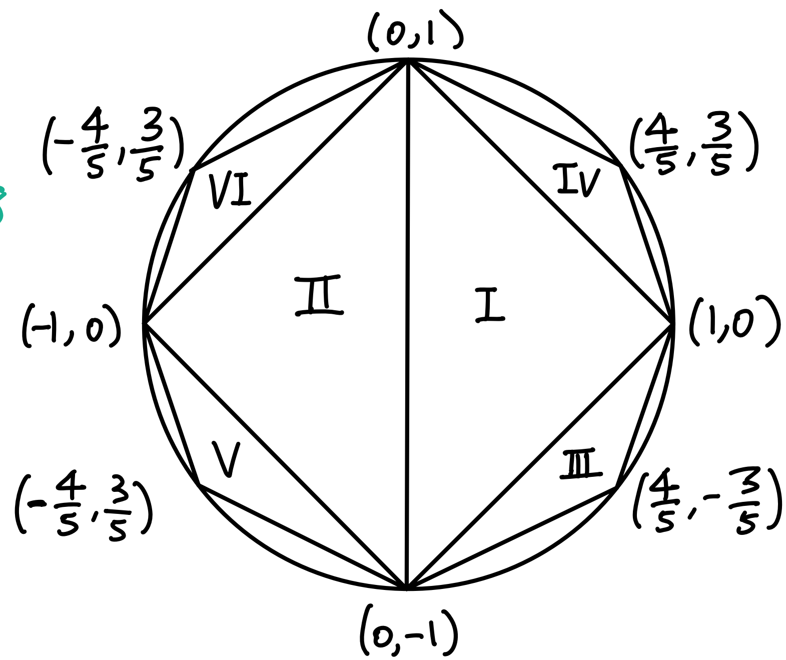
C'

i.e. a cone over $C' \times \{1\} \subseteq \mathbb{R}^3$

$C' = \{(x,y) \in \mathbb{R}^2 \mid x^2+y^2 < 1\}$

* Same as Voronoi decomposition 1908

* Cannizzo's ray corresponds to the point $(\frac{1}{2}, 0)$ in C'_{I}.



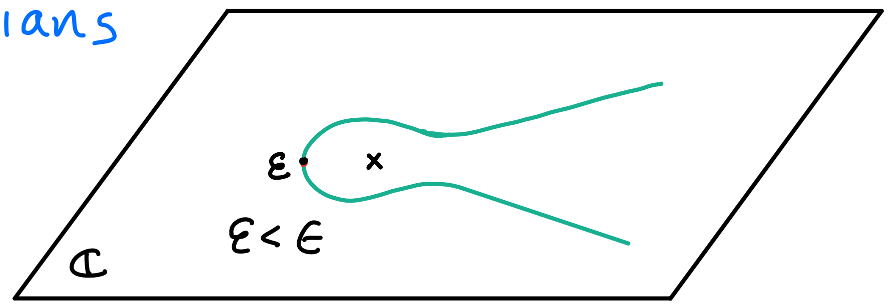
$GL(2,\mathbb{Z}) \ni h$ action: $\quad h\Omega = h\Omega h^T$

All chambers are in the same $GL(2,\mathbb{Z})$-orbit.

Homological Mirror Symmetry

$$\begin{tikzcd}
D^b\mathrm{Coh}(V_\tau) \arrow[rr, "L^*:\, \mathcal{L} \longmapsto \mathcal{L}|_{\Theta_\tau}"] \arrow[d, "{[\text{Polishchuk-Zaslow '98] } g=1}"', "{[\text{Fukaya 2002}]}"] & & D^b\mathrm{Coh}(\Theta_\tau) \arrow[d, "{[\text{Cannizzo 2020}]\ \tau = \frac{i}{2\pi}\log t \begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix},\ t>0 \text{ large}}", "{[\text{ACLL}]}"'] \\
H^0\mathrm{Fuk}_{\mathrm{aff}}(V_\tau^\vee) \arrow[rr, "U:\, \ell \mapsto U(\ell)"] & & H^0\mathrm{FS}_{\mathrm{aff}}(Y_\tau, W)
\end{tikzcd}$$

$W: Y_\tau \to \mathbb{C} \Rightarrow$
* horizontal distribution $(\ker dW)^{\perp\omega}$
* symplectic parallel transport $Y_{\gamma(0)} \to Y_{\gamma(1)}$
* fibered Lagrangians



HMS for the fiber abelian variety

[Polishchuk-Zaslow 1998] $g=1$
[Fukaya 2002]

$(\xi, \eta, \theta, \theta_\eta) \in Y_\tau \qquad \xi = (\xi_1, \dots, \xi_g), \quad \theta = (\theta_1, \dots, \theta_g)$

On a fiber $\ni (\xi, \theta)$ $\quad \eta =$ function of ξ, $\theta_\eta =$ constant

$$\omega = \sum_{k=1}^{g} d\xi_k \wedge d\theta_k = \sum_{j,k=1}^{g} \Omega_{jk}\, dr_j \wedge d\theta_k \qquad \xi = \Omega r$$

Complex side

$V_\tau^+ = \mathbb{C}^g / \mathbb{Z}^g + \tau \mathbb{Z}^g$

$\tau = B + i\Omega$

$\mathcal{L}_{k,[z]} := \mathcal{L}_\tau^{\otimes k} \otimes \mathbb{L}_{[z]}$

$z = a + \tau b \in V_\tau^+$, $a, b \in \mathbb{R}^g$

$V_\tau^+ \xrightarrow{\cong} \mathrm{Pic}^0(V_\tau^+)$, $[z] \mapsto \mathbb{L}_{[z]} = T_{[z]}^* \mathcal{L}_\tau \otimes \mathcal{L}_\tau^{-1}$

$T_{[z]} : V_\tau^+ \to V_\tau^+$, $[u] \mapsto [u+z]$

Symplectic side

$V_\tau^\vee \cong \mathbb{R}^{2g} / \mathbb{Z}^{2g} \ni (r, \theta)$

$$\omega_\tau^{\mathbb{C}} = \sum_{j,k=1}^{g} (B_{jk} + i\Omega_{jk})\, dr_j \wedge d\theta_k$$

$\hat{\ell}_{k,[z]} := (\ell_{k,[b]}, \mathcal{E}_{[a]})$

$[a], [b] \in (\mathbb{R}/\mathbb{Z})^g$

$\ell_{k,[b]} := \{ (r, \theta) \in \mathbb{R}^{2g}/\mathbb{Z}^{2g} \mid \theta = b - kr \}$

$\mathcal{E}_{[a]}$ trivial line bundle $\ell_{k,[b]} \times \mathbb{C}$
with flat $U(1)$ connection
$\nabla_{[a]} = d - 2\pi i\, a\, dr$

Fukaya-Seidel category of (Y_τ, W)

Generating objects: $\widehat{L}_{\mathbb{K},b} = (L_{\mathbb{K},b}, E_a)$

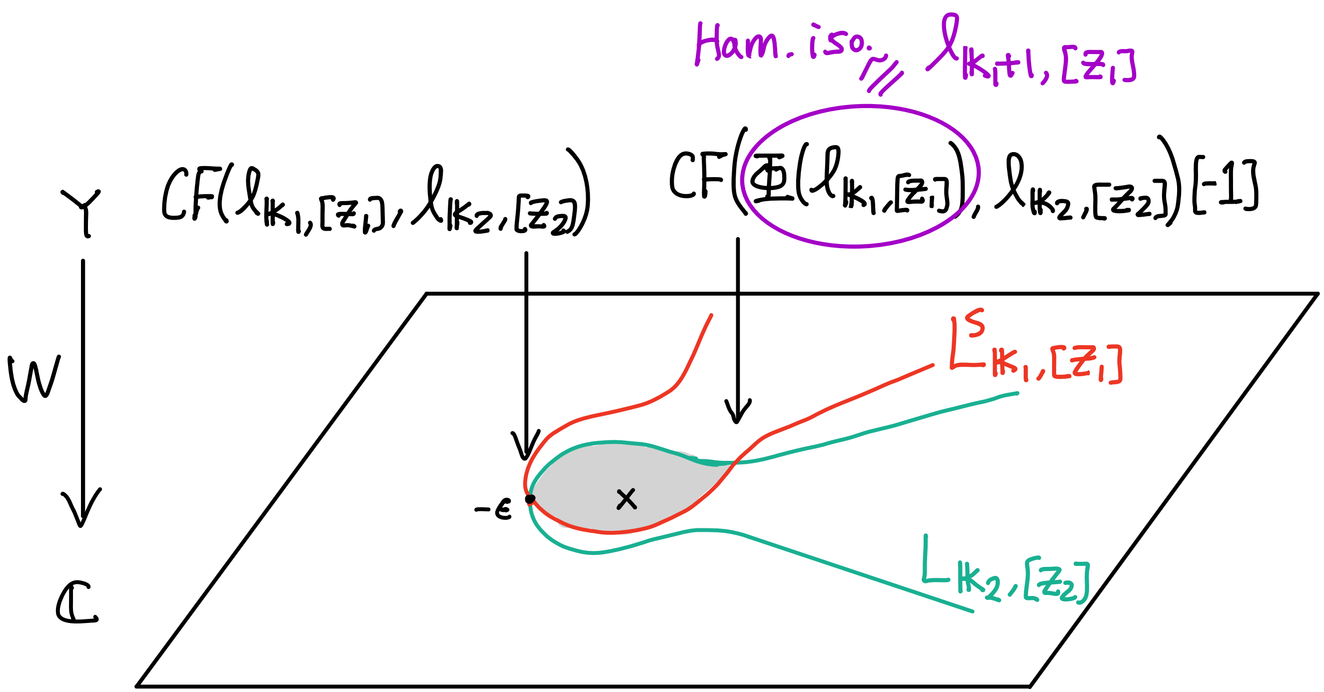
$$L_{\mathbb{K},b} := \bigcup_{t\in\mathbb{R}} \Phi^t_{\gamma_L}(\ell_{\mathbb{K},b})$$

$\Phi^t_{\gamma_L}$:= parallel transport along U-shaped $\gamma_L(t)$

$\ell_{\mathbb{K},b}$ linear Lagrangian in fiber $v_0^{-1}(\epsilon)$

E_a = trivial line bundle with $E_a|_{v_0^{-1}(\epsilon)} = \mathcal{E}_a$

equipped with $U(1)$ connection ∇_a with curvature $d\nabla_a = -2\pi i B|_{L_{\mathbb{K},b}}$



Φ = monodromy mirror to $- \otimes \mathcal{O}_{\Theta_\tau}$

Morphism

$\Delta\mathbb{K} = \mathbb{K}_2 - \mathbb{K}_1, \quad \Delta Z = Z_2 - Z_1$

$$\left(0 \longrightarrow \mathcal{L}_\tau^{-1} \longrightarrow \mathcal{O}_{V_\tau} \longrightarrow \mathcal{O}_{\Theta_\tau}\right) \otimes \mathcal{L}_{\Delta\mathbb{K},[\Delta Z]}$$

$$\begin{tikzcd}
H^0(V_\tau, \mathcal{L}_{\Delta\mathbb{K}-1,[\Delta Z]}) \arrow[d, phantom, "\shortparallel"] & H^0(V_\tau, \mathcal{L}_{\Delta\mathbb{K},[\Delta Z]}) \arrow[d, phantom, "\shortparallel"] & H^0(\Theta_\tau, \iota^*\mathcal{L}_{\Delta\mathbb{K},[\Delta Z]}) \arrow[d, phantom, "\shortparallel"] & \\
\mathrm{Hom}(\mathcal{L}_{\mathbb{K}_1+1,[Z_1]}, \mathcal{L}_{\mathbb{K}_2,[Z_2]}) \arrow[r, "\vartheta"] \arrow[d, "\cong"'] \arrow[dr, phantom, "\circlearrowright"] & \mathrm{Hom}(\mathcal{L}_{\mathbb{K}_1,[Z_1]}, \mathcal{L}_{\mathbb{K}_2,[Z_2]}) \arrow[r] \arrow[d, "\cong"] & \mathrm{Hom}(\mathcal{L}_{\mathbb{K}_1,[Z_1]}, \mathcal{L}_{\mathbb{K}_2,[Z_2]} \otimes \iota_*\mathcal{O}_{\Theta_\tau}) \arrow[r] \arrow[d] & 0 \\
CF(\underbrace{\hat{\ell}_{\mathbb{K}_1+1,[Z_1]}}, \hat{\ell}_{\mathbb{K}_2,[Z_2]})[-1] \arrow[r, "\partial"] & CF(\hat{\ell}_{\mathbb{K}_1,[Z_1]}, \hat{\ell}_{\mathbb{K}_2,[Z_2]}) \arrow[r] & HF(\hat{L}_{\mathbb{K}_1,[Z_1]}, \hat{L}_{\mathbb{K}_2,[Z_2]}) \arrow[r] & 0
\end{tikzcd}$$

Hamiltonian isotopic to $\Phi(\hat{\ell}_{\mathbb{K}_1,[Z_1]})$, Φ = monodromy

$$HF(\hat{L}_{\mathbb{K}_1,[Z_1]}, \hat{L}_{\mathbb{K}_2,[Z_2]}) = CF(\Phi(\ell_{\mathbb{K}_1,[Z_1]}), \hat{\ell}_{\mathbb{K}_2,[Z_2]})[-1] \oplus CF(\hat{\ell}_{\mathbb{K}_1,[Z_1]}, \hat{\ell}_{\mathbb{K}_2,[Z_2]})$$

$\partial = \cdot\,\vartheta(\tau, x)$ (up to a scale factor)

$\uparrow$ defining function of Θ_τ