Global HMS for Genus Two Curves

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Western Hemisphere Virtual Symplectic Seminar

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joint work with Haniya Azam

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Catherine Cannizzo's Thesis 2019

Abelian surface $\cong T^4$ genus 2 curve

$$
\begin{tikzcd}
\mathcal{L}^k \arrow[d, leftrightarrow] & D^b_{\mathcal{L}_\tau}Coh(V_\tau) \arrow[r, "\iota^*: \mathcal{L}^k \mapsto \mathcal{L}^k|_{\Theta_\tau}"] \arrow[d, "\text{Fukaya 2002}"'] \arrow[dr, phantom, "\circlearrowleft"] & D^b_{\mathcal{L}_\tau}Coh(\Theta_\tau) \arrow[d, "\text{Cannizzo 2020}"] \\
\ell_k & H^0 Fuk(V_\tau^\vee) \arrow[r, "\cup"] & H^0 FS(Y_\tau, v_0)
\end{tikzcd}
$$

$\ell_k \rightsquigarrow \cup(\ell_k)$

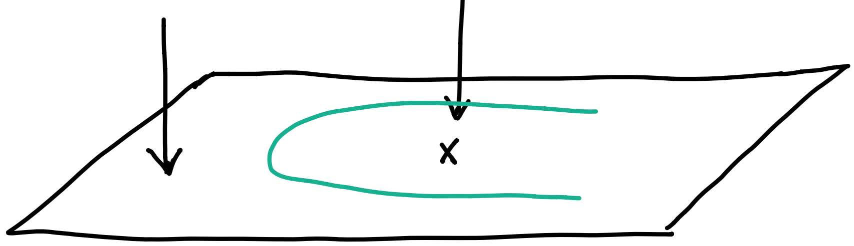
Mirror construction based on SYZ, AAK $Bl_{\Theta_\tau \times \{0\}} V_\tau \times \mathbb{C}$

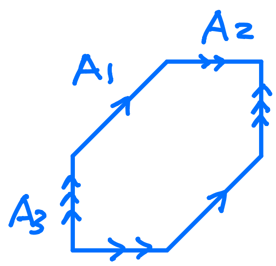
Symplectic fibration

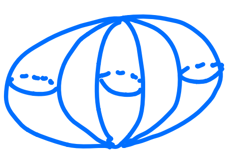
$$Y_\tau \xrightarrow{v_0} \mathbb{C}$$

generic fiber $V^\vee \simeq T^4$

Singular fiber $\mathbb{CP}^2(3)/\Omega\mathbb{Z}^2$







$\mathrm{Crit}\, v_0 = C_1 \cup C_2 \cup C_3 \subseteq Y$

$C_j \cong \mathbb{CP}^1$ with symplectic area A_j

Genus two curve for any complex parameter

$$\tau = \begin{pmatrix} \tau_{11} & \tau_{12} \\ \tau_{21} & \tau_{22} \end{pmatrix} = B + i\Omega, \quad \text{symmetric}, \qquad \det \Omega > 0$$

In Cannizzo's thesis, $\tau = \frac{i}{2\pi} \log t \begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix}$, t large.

$$\Theta_\tau = \left\{ \vartheta(x_1, x_2; \tau) = \sum_{n=(n_1,n_2)\in\mathbb{Z}^2} x_1^{n_1} x_2^{n_2} e^{\pi i n^T \tau n} = 0 \right\} \Big/ \tau\mathbb{Z}^2$$

$$\cap\!|$$

$$V_\tau = (\mathbb{C}^*)^2 / \tau\mathbb{Z}^2 = \left\{ (x_1, x_2) \in (\mathbb{C}^*)^2 \right\} \Big/ (x_1,x_2) \sim (\tau n)\cdot(x_1,x_2)$$

$$\tau\begin{pmatrix} n_1 \\ n_2 \end{pmatrix} \cdot (x_1, x_2) = \begin{pmatrix} \tau_{11} n_1 + \tau_{12} n_2 \\ \tau_{21} n_1 + \tau_{22} n_2 \end{pmatrix} \cdot (x_1, x_2)$$

$$= \left(e^{2\pi i(\tau_{11} n_1 + \tau_{12} n_2)} x_1, \; e^{2\pi i(\tau_{21} n_1 + \tau_{22} n_2)} x_2 \right)$$

$$\Big\uparrow \wr \begin{pmatrix} x_1 = e^{2\pi i v_1} \\ x_2 = e^{2\pi i v_2} \end{pmatrix}$$

$$V_\tau^+ = \mathbb{C}^2 / \mathbb{Z}^2 + \tau\mathbb{Z}^2 = \left\{ (v_1, v_2) \in \mathbb{C}^2 \right\} / \mathbb{Z}^2 + \tau\mathbb{Z}^2$$

Mirror to abelian surface V_τ

$$V_\tau^\vee \cong \mathbb{R}^4/\mathbb{Z}^4, \quad \omega_{V^\vee}^{\mathbb{C}} = \left(\sum_{j,k=1}^{2} B_{jk}\, d\rho_j \wedge d\theta_k \right) + i \underbrace{\left(\sum_{j,k=1}^{2} \Omega_{jk}\, d\rho_j \wedge d\theta_k \right)}_{= \sum_{k=1}^{2} d\xi_k \wedge d\theta_k \ (\text{for } \xi = \Omega\rho)}$$

$$\rotatebox{90}{$\in$}$$

$$(\rho, \theta)$$

$$l_K = \left\{ (\rho, \theta) \in V_\tau^\vee \mid \theta = -K\rho \right\}$$

Mirror Landau-Ginzburg model (Y, v_0)

$(x_1, x_2) \longmapsto \xi = (\xi_1, \xi_2) = (\log|x_1|, \log|x_2|)$

$$\begin{tikzcd}
\widetilde{\Theta}_\tau \arrow[r, hook] \arrow[d] & (\mathbb{C}^*)^2 \arrow[r] \arrow[d] & \mathbb{R}^2 \arrow[d] \\
\Theta_\tau \arrow[r, hook] & V_\tau = (\mathbb{C}^*)^2/\tau\mathbb{Z}^2 \arrow[r] & \mathbb{R}^2/\Omega\mathbb{Z}^2
\end{tikzcd}$$

$\rightsquigarrow$ [Abouzaid-Auroux-Katzarkov]

$k(n) = -\frac{1}{2} n^T \Omega n$

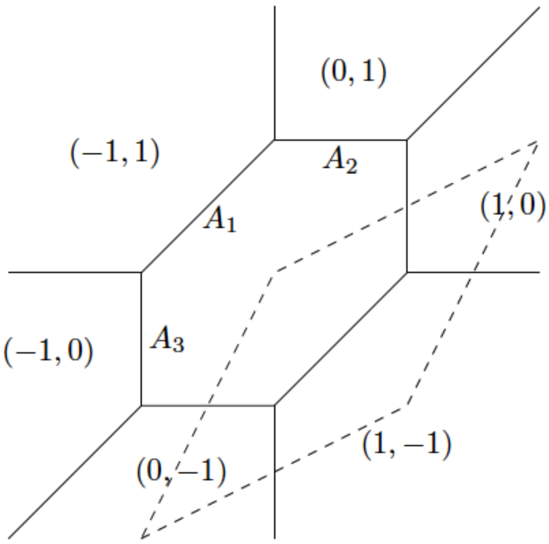
$$\Delta_{\widetilde{Y}_\tau} = \left\{ (\xi, \eta) \in \mathbb{R}^3 \;\middle|\; \eta \geq \varphi(\xi) = \max_{n \in \mathbb{Z}^2} \{ \langle \xi, n \rangle + k(n) \} \right\}$$

$$= \bigcap_{n \in \mathbb{Z}^2} \left\{ (\xi, \eta) \in \mathbb{R}^3 \;\middle|\; L_n(\xi, \eta) = -n_1 \xi_1 - n_2 \xi_2 + \eta - k(n) \geq 0 \right\}$$

$v_n = \begin{pmatrix} -n_1 \\ -n_2 \\ 1 \end{pmatrix}$ normal to facet $\{L_n(\xi, \eta) = 0\}$

$\Delta_{Y_\tau} = \Delta_{\widetilde{Y}_\tau} / \Omega\mathbb{Z}^2$ (quotient near singular fiber of $v_0 : \widetilde{Y} \to \mathbb{C}$)

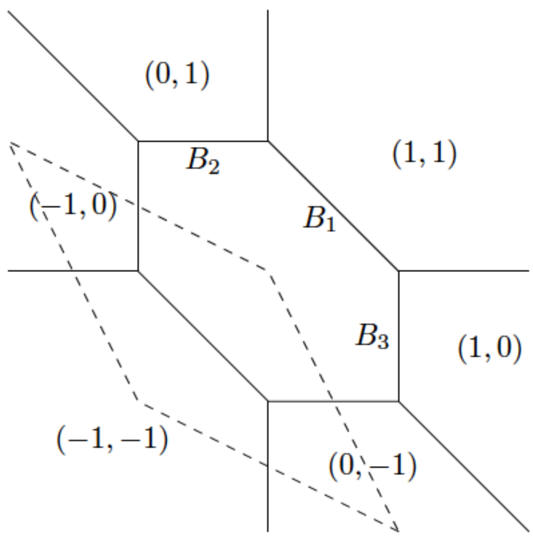
$$(\Omega m) \cdot (\xi_1, \xi_2, \eta) = \left(\xi_1 + (\Omega m)_1, \, \xi_2 + (\Omega m)_2, \, \eta - k(m) + \langle \xi, m \rangle \right)$$



Case 1:

$$\begin{cases} \Omega_{12} = \Omega_{21} > 0 \\ \Omega_{11} - \Omega_{12} > 0 \\ \Omega_{22} - \Omega_{12} > 0 \end{cases}$$

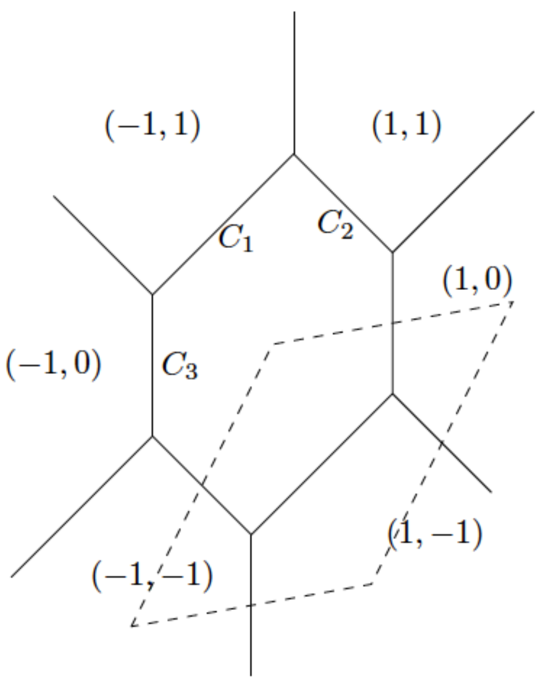
$$\begin{pmatrix} \Omega_{11} & \Omega_{12} \\ \Omega_{21} & \Omega_{22} \end{pmatrix} = \begin{pmatrix} A_1 + A_2 & A_1 \\ A_1 & A_1 + A_3 \end{pmatrix}$$



Case 2:

$$\begin{cases} \Omega_{12} = \Omega_{21} < 0 \\ \Omega_{11} - \Omega_{12} > 0 \\ \Omega_{22} - \Omega_{12} > 0 \end{cases}$$

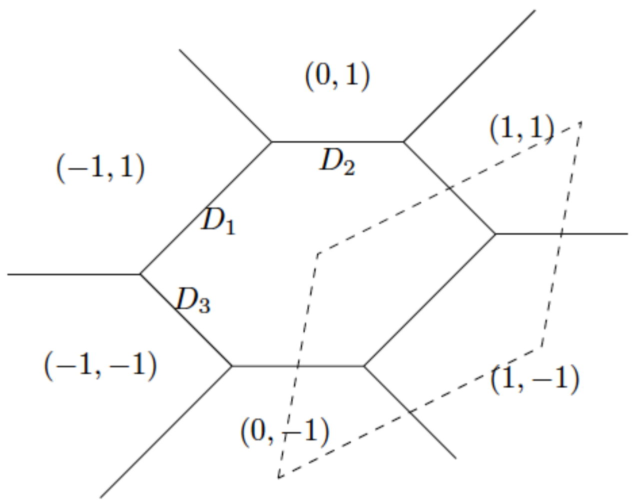
$$\begin{pmatrix} \Omega_{11} & \Omega_{12} \\ \Omega_{21} & \Omega_{22} \end{pmatrix} = \begin{pmatrix} B_1 + B_2 & -B_1 \\ -B_1 & B_1 + B_3 \end{pmatrix}$$



Case 3:

$$\begin{cases} \Omega_{12} = \Omega_{21} > 0 \\ \Omega_{11} - \Omega_{12} < 0 \\ \Omega_{22} - \Omega_{12} > 0 \end{cases}$$

$$\begin{pmatrix} \Omega_{11} & \Omega_{12} \\ \Omega_{21} & \Omega_{22} \end{pmatrix} = \begin{pmatrix} C_1 + C_2 & \frac{C_1 - C_2}{2} \\ \frac{C_1 - C_2}{2} & C_3 \end{pmatrix}$$

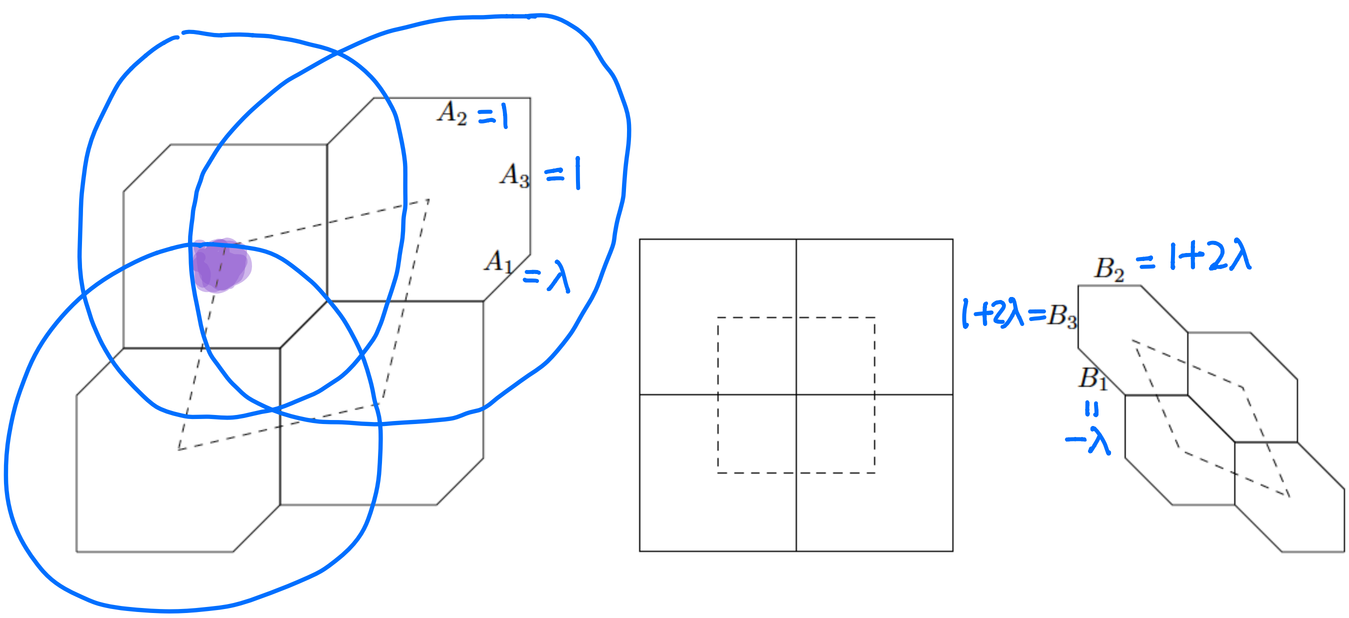


Case 4:

$$\begin{cases} \Omega_{12} = \Omega_{21} > 0 \\ \Omega_{11} - \Omega_{12} > 0 \\ \Omega_{22} - \Omega_{12} < 0 \end{cases}$$

$$\begin{pmatrix} \Omega_{11} & \Omega_{12} \\ \Omega_{21} & \Omega_{22} \end{pmatrix} = \begin{pmatrix} D_2 & \frac{D_1 - D_3}{2} \\ \frac{D_1 - D_3}{2} & D_1 + D_3 \end{pmatrix}$$

<u>Example</u> $\quad \Omega = \begin{pmatrix} 1+\lambda & \lambda \\ \lambda & 1+\lambda \end{pmatrix}$



$\lambda > 0$ $\qquad$ $\lambda = 0$ $\qquad$ $\lambda < 0$

Symplectic structure on Y

of Guillemin, Kanazawa-Lau

On $\tilde{Y}$:

$$\tilde{G}(\xi,\eta) = \sum_{n\in\mathbb{Z}^2} \chi_n(\xi,\eta)\, L_n(\xi,\eta) \log L_n(\xi,\eta)$$

$$\Psi^{jk} = \frac{\partial^2 G}{\partial \xi_j \partial \xi_k} = \frac{\partial \rho_j}{\partial \xi_k}, \qquad j,k = 1,2,\ 3:=\eta \ ,\quad \xi_\eta := \eta$$

$$\rho_j = \frac{\partial \tilde{G}}{\partial \xi_j}$$

$$\xi_j = \frac{\partial \tilde{F}}{\partial \rho_j}$$

$$\Psi_{jk} = \left(\Psi^{jk}\right)^{-1} = \frac{\partial \xi_j}{\partial \rho_k} = \frac{\partial^2 \tilde{F}}{\partial \rho_j \partial \rho_k}$$

$$\omega = \sum_{j,k=1}^{3} \Psi_{jk}\, d\rho_j \wedge d\theta_k = \frac{i}{2}\sum_{j,k=1}^{3} \Psi_{jk}\, du_j \wedge d\bar{u}_k = \sum_{k=1}^{3} d\xi_k \wedge d\theta_k$$

Complex toric coordinates

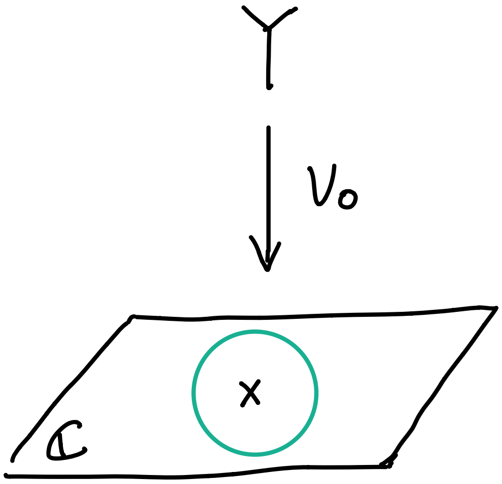
$$\left(t_1 = e^{u_1},\ t_2 = e^{u_2},\ t_3 = e^{u_3}\right) \in (\mathbb{C}^*)^3$$

$$u_j = \rho_j + i\theta_j$$

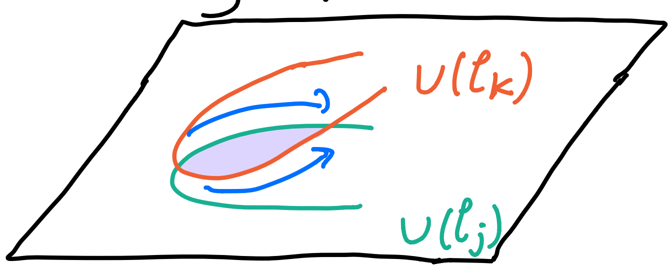
Monodromy

Proposition: $\varphi_{2\pi}(\ell_k)$ is Hamiltonian isotopic to ℓ_{k+1}

Monodromy, i.e. parallel transport along a circle re^{it} in the base $\mathbb{C}$



Proposition useful for: Computing differentials



Parallel transport

* $V_0 = t_3 = e^{\rho_\eta + i\theta_\eta} = e^{\frac{\partial G}{\partial \eta}(\xi,\eta)} e^{i\theta_\eta}$

* Horizontal lift $\left(\frac{\partial}{\partial\theta}\right)^{\#}$ of $\frac{\partial}{\partial\theta}$ is in the span of the Hamiltonian vector field generated by $\log|v|$

$$d(\log|v|) = \Psi^{\eta 1} d\xi_1 + \Psi^{\eta 2} d\xi_2 + \Psi^{\eta\eta} d\eta$$

$$\left(\frac{\partial}{\partial\theta}\right)^{\#} = \frac{\partial}{\partial\theta_\eta} + \frac{\Psi^{\eta 1}}{\Psi^{\eta\eta}}(\xi,\eta)\frac{\partial}{\partial\theta_1} + \frac{\Psi^{\eta 2}}{\Psi^{\eta\eta}}(\xi,\eta)\frac{\partial}{\partial\theta_2}$$

* Parallel transport is

$$\varphi_t(\xi_1,\xi_2,\eta,\theta_1,\theta_2,\theta_\eta)$$

$$= \left(\xi_1,\xi_2,\eta,\ \theta_1 + \frac{\Psi^{\eta 1}}{\Psi^{\eta\eta}}(\xi,\eta)t,\ \theta_2 + \frac{\Psi^{\eta 2}}{\Psi^{\eta\eta}}(\xi,\eta)t,\ \theta_\eta + t\right)$$

* Proof of the proposition :

- On a fiber $v_0^{-1}(-r)$, has an Hamiltonian vector field

$$X = \sum_{j=1}^{2}\left(\underbrace{\frac{\Psi^{\eta j}}{\Psi^{\eta\eta}}(\xi,\eta)}_{=:f_j(\xi)} + \sum_{l=1}^{2}\Omega^{jl}\xi_l\right)\frac{\partial}{\partial\theta_j}.$$

s.t. $\phi_X^{2\pi}(l_{k+1}) = \varphi_{2\pi}(l_k)$

- ϕ_X^t is well-defined on the quotient $Y = \tilde{Y}/\Omega\mathbb{Z}^2$

Reason: moment map is $\Omega\mathbb{Z}^2$-equivariant

can show $f_j(\xi+\Omega m) = f_j(\xi) - m$.

* Extra fun fact:

Fix $b>0$, consider

$$R_n(b) := \{ \xi \in \mathring{\Delta}_{\tilde{Y}} \mid L_m(\xi,\eta) \geq b \text{ if } m \neq n \}$$

As $L_n(\xi,\eta) \to 0$ on $R_n(b)$, we have

$$\frac{\Psi^{\eta j}}{\Psi^{\eta\eta}}(\xi,\eta) \longrightarrow \begin{pmatrix} -n_1 \\ -n_2 \end{pmatrix}$$

normal vector
fits intuition