# Counting special Lagrangian classes and semistable Mukai vectors for K3 surfaces

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Moduli Across the Pandemic

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Part I : Motivation and Problems

Part II : Results

# Flat sufaces

holomorphic 1-form $\Omega$ on Riemann surface $\Sigma_1$ $\rightsquigarrow$ $S = (\Sigma_1, g = \frac{1}{2}\Omega\bar{\Omega})$ flat surface

flat metric with conical singularity at zeros of $\Omega$

**Geodesics** $\gamma \in \Sigma_1$ s.t. $\mathrm{Im}(e^{i\phi}\Omega)|_\gamma = 0$

(locally $\Omega = dz$, $\mathrm{Im}(e^{i\phi}dz)|_\gamma = 0$ $\Rightarrow$ $\gamma$ straight line of angle $\phi$

**Counting problems** (normalize area to 1, area form $= \frac{i}{2}\Omega\wedge\bar{\Omega}$)

$N_{sc}(S, L)$ = # of saddle connections of length at most $L$.

$N_{cg}(S, L)$ = # of maximal cylinders filled with closed geodesics of length at most $L$.

[Mazur 1990] for all flat surfaces, $C_1(S)L^2 \leq N(S,L) \leq C_2(S)L^2$

[Eskin-Masur 2001] for almost all flat surfaces

$$\lim_{L\to\infty} \frac{N_{sc}(S,L)}{L^2} = \mathrm{Const}_{sc}, \quad \lim_{L\to\infty} \frac{N_{cg}(S,L)}{L^2} = \mathrm{Const}_{cg}$$

[Eskin-Mirazkahani-Mohammadi 2015] Cesàro-type quadratic asymptotics for all flat surfaces.

# Beyond flat surfaces

| Flat surfaces | Calabi-Yau manifold | Triangulated categories |
|---|---|---|
| holomorphic 1-form | holomorphic top-form | Stability conditions |
| geodesics | Special Lagrangians | Semistable objects |
| Length | Period integral | Central charge |

## Calabi-Yau manifold $(X, \Omega, \omega)$

Ricci-flat Kähler form

$\omega$ defines Lagrangian submanifold : $\omega|_L = 0$

$\omega, \Omega$ defines special Lagrangian submanifold (sLag): $\omega|_L = 0$

$\mathrm{Im}(e^{i\phi}\Omega)|_L = 0$ (phase $\phi$)

Period integral : $Z(L) = \int_L \Omega$

## Counting problem

if $\dim_{\mathbb{C}} X = n$

$$SL_{\omega,\Omega}(R) = \# \left\{ \gamma \in H^n(X, \mathbb{Z}) \ : \ \exists \text{ irreducible sLag } L \text{ s.t. } [L]^{PD} = \gamma, \ |\gamma.\Omega = \int_L \Omega| \leq R \right\}$$

# Triangulated categories and mirror symmetry

For a mirror pair of Calabi-Yau manifolds $(X, \omega_X, J_X)$, $(Y, \omega_X, J_Y)$

Homological mirror symmetry :

$$D^\pi \mathrm{Fuk}(X, \omega_X) \cong D^b \mathrm{Coh}(Y, J_Y) \quad \text{and} \quad D^b \mathrm{Coh}(X, \omega_X) \cong D^\pi \mathrm{Fuk}(Y, \omega_Y)$$

Fukaya category   objects : Lagrangian submanifolds

stable objects : special Lagrangian submanifolds

$D^b\mathrm{Coh}$   objects : coherent sheaves (defined by $J$)

stable objects : stable coherent sheaves (defined by $J, \omega$)

E.g. $E$ holomorphic vector bundle on a complex curve

Slope $\mu(E) := \deg E / \mathrm{rk}\, E$

$E$ stable (semistable) if every subbundle $F$ satisfies

$$\mu(F) < \mu(E) \quad (\mu(F) \leq \mu(E))$$

## K3 surface

A compact complex surface that admits a nowhere vanishing holomorphic 2-form $\Omega$ and is simply connected. [Siu83] all K3 are Kähler

**Cohomology** all K3 surfaces are diffeomorphic

$$H^0(X,\mathbb{Z}) \cong H^4(X,\mathbb{Z}) \cong \mathbb{Z}, \quad H^2(X,\mathbb{Z}) \cong \mathbb{Z}^{22}, \quad H^1(X,\mathbb{Z}) = H^3(X,\mathbb{Z}) = 0$$

**K3 lattice** $H^2(X,\mathbb{Z})$, $\underbrace{(-,-)}_{\text{Intersection pairing}} : H^2(X,\mathbb{Z}) \otimes H^2(X,\mathbb{Z}) \longrightarrow \mathbb{Z}$

signature $= (3,19)$

**Weight-two Hodge structure** $H^2(X,\mathbb{C}) = H^{2,0}(X) \oplus H^{1,1}(X) \oplus H^{0,2}(X)$

$$h^{2,0} = 1 \qquad h^{1,1} = 20 \qquad h^{0,2} = 1$$

**Néron-Severi lattice** $NS(X)$ = isomorphism classes of line bundles classified by $c_1 : Pic(X) \longrightarrow H^2(X,\mathbb{Z})$

For K3, $Pic(X) = NS(X) = H^{1,1}(X,\mathbb{Z}) := H^{1,1}(X) \cap \mathrm{Image}\,(H^2(X,\mathbb{Z}) \longrightarrow H^2(X,\mathbb{C}))$

$\rho = rk(Pic(X))$, $\quad 0 \leq \rho \leq 20$

$(-,-) : Pic(X) \otimes Pic(X) \longrightarrow \mathbb{Z}$, $\quad (L, L') = \int_X c_1(L) \wedge c_1(L')$

For projective K3, $1 \leq \rho \leq 20$, $\quad (-,-)$ has signature $(1, \rho - 1)$

# Coherent sheaves and Mukai vectors

$X$ = algebraic/projective K3 surface

$D = D^b\mathrm{Coh}(X)$

$K(D)$ = Grothendieck group

$$\begin{tikzcd} A \arrow[rr] & & B \arrow[dl] \\ & C \arrow[ul, "{[1]}"] & \end{tikzcd} \Rightarrow [B]=[A]+[C] \text{ in } K(D)$$

**Mukai vector** $\quad v : K(D) \longrightarrow H^*(X,\mathbb{Z}) = H^0(X,\mathbb{Z}) \oplus H^2(X,\mathbb{Z}) \oplus H^4(X,\mathbb{Z})$

$$v(E) := ch(E)\sqrt{td(X)} = \big(rk(E),\ c_1(E),\ \chi(E) - rk(E)\big)$$

**Mukai pairing** $\quad \langle -,- \rangle : H^*(X,\mathbb{Z}) \longrightarrow H^*(X,\mathbb{Z})$

$$\langle (r_1, D_1, s_1),\ (r_2, D_2, s_2) \rangle = D_1 . D_2 - r_1 . s_2 - r_2 . s_1$$

$$v : \big(K(D),\ -\chi(-,-)\big) \longrightarrow \underbrace{\big(H^*(X,\mathbb{Z}),\ \langle -,- \rangle\big)}_{\text{Mukai lattice}}$$

Euler pairing

**Mukai lattice** signature $(4,20)$

$$\chi(E,F) := \sum_k (-1)^k \dim \mathrm{Hom}_D^k(E,F)$$

**Numerical Grothendieck group** $\quad N(D) = K(D) / \ker \chi(-,-)$

$$\big(N(D),\ -\chi(-,-)\big) \overset{v}{\cong} \big(H^0(X,\mathbb{Z}) \oplus NS(X) \oplus H^4(X,\mathbb{Z}),\ \langle -,- \rangle\big)$$

Signature $(2, \rho)$, $\rho = rk(NS(X))$

# Bridgeland stability conditions : definition

$\mathrm{Stab}(\mathcal{D}) \ni \sigma = (Z, \mathcal{P})$ locally finite numerical Bridgeland stability condition

a complex manifold

* $Z : N(\mathcal{D}) \longrightarrow \mathbb{C}$ central charge, a group homomorphism
* $\mathcal{P} := \{\mathcal{P}(\phi)\}_{\phi \in \mathbb{R}}$, $\mathcal{P}(\phi)$ = semistable objects of phase $\phi$.

Satisfying the following axioms

(1) $E \in \mathcal{P}(\phi) \Rightarrow Z(E) \in \mathbb{R}_{>0}\, e^{i\pi\phi}$

(2) $\phi_1 > \phi_2$, $E_j \in \mathcal{P}(\phi_j)$, $j=1,2 \Rightarrow \mathrm{Hom}(E_1, E_2) = 0$.

(3) $\mathcal{P}(\phi+1) = \mathcal{P}(\phi)[1]$

(4) (Harder-Narasimhan filtration) for each $0 \neq E \in \mathcal{D}$, there exists

$$\begin{tikzcd}[column sep=small]
0 = E_0 \arrow[rr] & & E_1 \arrow[rr] \arrow[dl] & & E_2 \arrow[r] \arrow[dl] & \cdots \arrow[r] & E_{K-1} \arrow[rr] & & E \arrow[dl] \\
& B_1 \arrow[ul] & & B_2 \arrow[ul] & & & & B_K \arrow[ul] &
\end{tikzcd}$$

for $B_j \in \mathcal{P}(\phi_j)$

$\phi_1 > \phi_2 > \cdots > \phi_K$

(5) (Support Property) $\exists$ constant $b>0$ and a norm $\|\cdot\|$ on $N(\mathcal{D}) \otimes_{\mathbb{Z}} \mathbb{R}$ s.t. for any semistable object $E$, we have $\|E\| \le C|Z(E)|$

## Counting function

$$N_\sigma(R) = \#\{\gamma \in N(\mathcal{D}) : \exists \text{ a } \sigma\text{-semistable object } E \text{ with } v(E) = \gamma,\ |Z_\sigma(\gamma)| \le R\}$$

# Special Lagrangian classes

**Lagrangian class lattice** $\mathrm{Lag}(X,\omega) := \{\gamma \in H^2(X,\mathbb{Z}) \,,\, \gamma = [L]^{Pd}\}$

[Schoen-Wolfson 2001] $= H^2(X,\mathbb{Z}) \cap \omega^{\perp} \leq H^2(X,\mathbb{Z})$

Intersection pairing

$\mathrm{Slag}(X,\omega,\Omega) := \{\gamma \in H^2(X,\mathbb{Z}) : \exists$ irreducible sLag $L$ with $[L]^{Pd} = \gamma\}$

[Lai-Lin-Schaffler] $\subseteq \{\gamma \in \mathrm{Lag}(X,\omega) : \gamma^2 \geq -2\}$

**Counting problem** $SL_{\omega,\Omega}(R) = \#\{\gamma \in \mathrm{SLag}(X,\omega,\Omega) \,,\, |\gamma . \Omega| \leq R\}$

**Fukaya category** $F := D^{\pi}\mathrm{Fuk}(X)$

$K(F) \xrightarrow{ch} HH_0(F) \xrightarrow{OC} H^2(X,\Lambda) \,,\quad [L] \longmapsto [L]^{Pd} \in \mathrm{Lag}(X,\omega)$ when $L$ geometric

open-closed map

[Sheridan-Smith 2020]

$\langle ch(L_1), ch(L_2)\rangle = -\chi(L_1,L_2) = -\chi(HF^*(L_1,L_2)) = [L_1].[L_2]$

[Shklyanov 2013]

$N(F) := K(F)/_{\ker \chi(-,-)}$

<u>Mirror symmetry</u> $\Rightarrow$ $N(F(X)) \cong N(D(Y))$

not know whether $N(F(X)) = \mathrm{Lag}(X,\omega)$

## Twistor sphere

$X$ = K3 surface $\Rightarrow$ Hyperkähler
$g$ = Ricci-flat metric
$P \subseteq H^2(X, \mathbb{R})$ any positive definite 3-plane

**Twistor family** there is a 2-sphere family $(X, J_t)$, $t \in \mathbb{S}^2$
all compatible with $g$
$\omega_t \in \mathbb{S}^2(P)$

**Counting problem**

$$SL_P(R) = \#\{\gamma \in H^2(X, \mathbb{Z}) : \exists\, \omega_t \in \mathbb{S}^2(P),\ \gamma \in SLag(X, \omega_t, \Omega_t),\ |\gamma . \Omega_t| \leq R\}$$

[Filip 2020] studies count of special Lagrangian tori in this twistor sphere formulation.

[Kachru-Tripathy-Zimet 2020]

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# Bridgeland stability conditions : properties

Theorems by Bridgeland 2007, 2008

$*$ $\mathrm{Stab}(\mathcal{D}) \circlearrowleft \mathrm{Aut}(\mathcal{D})$

$\cup$

$\mathrm{Stab}^{+}(\mathcal{D}) \leftarrow$ a connected component

$\exists\, \phi \in \mathrm{Aut}(\mathcal{D})$ s.t. $\phi : \mathrm{Stab}^{+}(\mathcal{D}) \longrightarrow \overline{\mathcal{U}(\mathcal{D})}$

$\cup$

$\Rightarrow$ for almost all $\sigma \in \mathrm{Stab}^{+}(\mathcal{D})$, $\exists\, \sigma' \in \mathcal{U}(\mathcal{D})$ s.t. $N_{\sigma}(\mathbb{R}) = N_{\sigma}(\mathbb{R})$

$\mathcal{U}(\mathcal{D}) \leftarrow$ geometric stability conditions : $\sigma \in \mathrm{Stab}(X)$ s.t. all $\mathcal{O}_x$, $x \in X$, are $\sigma$-stable of the same phase

$\cup$ for $\sigma \in \mathcal{U}(\mathcal{D})$, $Z_{\sigma} = Z_{\sigma'} g$ for $\sigma' \in V(\mathcal{D})$, $g \in GL^{+}(2, \mathbb{R})$.

$V(\mathcal{D}) \leftarrow$ geometric stability condition of phase 1 constructed via tilting.

For $\sigma \in V(\mathcal{D})$, $Z_{\sigma}(v) = \langle \exp(B + i\omega), v \rangle$

$$= \left\langle \left(1, B, \frac{B^2 - \omega^2}{2}\right), v \right\rangle + i \left\langle (0, \omega, B.\omega), v \right\rangle$$

$B$, $\omega$ are real divisor classes, $\omega$ ample

**Theorem [Athreya-Fan-L.]** For almost every $\sigma \in \mathrm{Stab}^{+}(\mathcal{D})$, we have

$$N_\sigma(R) = C(\sigma) R^{\rho+2} + o(R^{\rho+2}) \qquad \rho = \mathrm{rk}(\mathrm{Pic}(X))$$

For $\sigma \in V(\mathcal{D})$, $\displaystyle C(\sigma) = \frac{2\pi^{(\rho+2)/2}}{(\rho+2)\,\Gamma(\frac{\rho}{2}+1)\,(\omega^2)^{(\rho+2)/2}\sqrt{\mathrm{Disc}\, NS(X)}}$.

For $\sigma \in U(\mathcal{D})$, $Z_\sigma = Z_{\sigma'} g$ for $\sigma' \in V(\mathcal{D})$, $g \in GL^{+}(2,\mathbb{R})$.

<u>Case 1</u> $g \in \mathbb{R}^{+}$, then $\displaystyle C(\sigma) = \frac{C(\sigma')}{g^{\rho+2}}$

<u>Case 2</u> $g$ in the rotation part of $SL(2,\mathbb{Z})$, then $C(\sigma) = C(\sigma')$.

<u>Case 3</u> $g = \begin{pmatrix} 1 & \kappa \\ 0 & \lambda \end{pmatrix}$ is a shear by $\kappa + i\lambda$, $\lambda > 0$, then

$$C(\sigma) = \frac{\pi^{\rho/2} \int_0^{2\pi} \left(\cos^2\theta + \frac{1}{\lambda^2}(\sin\theta - \kappa\cos\theta)^2\right)^{\rho/2} d\theta}{(\rho+2)\,\Gamma(\frac{\rho}{2}+1)\,\lambda\,(\omega^2)^{(\rho+2)/2}\sqrt{\mathrm{Disc}\; NS(X)}}$$

**Remark** $C(\sigma)$ depends on : * rank and discriminant of $NS(X)$
* $\omega^2$ ( $B$ does not matter)

# Reason

**Theorem [Bayer-Macri 2014]**

For $v = m v_0 \in N(\mathcal{D})$, $v_0$ primitive, $m \in \mathbb{Z}_+$, $\sigma \in \mathrm{Stab}^\dagger(\mathcal{D})$ $\underbrace{\text{generic}}$, avoid walls (measure 0 set)

then $v = v[E]$, for some $E$ semistable

$\iff v_0^2 \geq -2$

$$N_\sigma(R) = \#\Big\{ v \in N(\mathcal{D}) : v = m v_0,\ m \in \mathbb{Z}_+,\ v_0 \text{ primitive},\ \underbrace{v_0^2 \geq -2}_{\text{indefinite}},\ \underbrace{|Z_\sigma(v)|^2 \leq R}_{\text{positive semidefinite}} \Big\}$$

$$= \boxed{\#\{ v \in N(\mathcal{D}) : v^2 \geq 0,\ |Z_\sigma(v)| \leq R \}}$$

Mukai lattice even $\Rightarrow$ no $v^2 = -1$

$$+ \boxed{\#\{ v \in N(\mathcal{D}) : v = m v_0,\ m \in \mathbb{Z}_+,\ v_0^2 = -2,\ |Z_\sigma(v)| \leq R \}}$$

$\boxed{\quad}$

[Fan 2021] $\mathrm{sys}(\sigma) := \min\{ |Z_\sigma(v(E))| : E \text{ is a } \sigma\text{-semistable object} \}$

$\Rightarrow$ if $v = m v_0$, $|Z_\sigma(v)| < R$ $\Rightarrow$ $m < \dfrac{R}{\mathrm{sys}(\sigma)}$

$$\Rightarrow \boxed{\quad} < \frac{R}{\mathrm{sys}(\sigma)} \cdot \underbrace{\#\{ v \in N(\mathcal{D}) : v^2 = -2,\ |Z_\sigma(v)| \leq R \}}_{\sim R^\rho \text{ by [Duke-Rudnick-Sarnak 1993]}}$$

$$= o(R^{\rho+2})$$

$$\Upsilon_\sigma(R) = \{ v \in N(D)_{\mathbb{R}} \cong \mathbb{R}^{\rho+2},\ v^2 \geq 0,\ |Z_\sigma(v)|^2 \leq R^2 \}$$

$$\Upsilon_\sigma(R) = \{ Rv \mid v \in \Upsilon_\sigma(1) \} = R \cdot \Upsilon_\sigma(1)$$

(Gauss circle problem) $\displaystyle\lim_{R\to\infty} \frac{\boxed{\#\{ v \in \mathbb{Z}^{\rho+2} \cap \Upsilon_\sigma(R) \}}}{R^{\rho+2}} = \mathrm{Vol}(\Upsilon_\sigma(1)).$

**Theorem [Athreya-Fan-L.]** Assuming $\mathrm{Lag}(X,\omega)$ has signature $(2,19)$, then

$$SL_{\omega,\Omega}(R) \leq C(\omega,\Omega) R^{21} + o(R^{21})$$

where

$$C(\omega,\Omega) = \frac{2\pi^{21/2}}{21\, \Gamma(\frac{21}{2})\, K_\Omega^{21/2} \sqrt{\mathrm{Disc}\, \mathrm{Lag}(X,\omega)}}$$

if $\omega$ a rational Kähler class
so statement holds for polarized K3

$K_\Omega = (\mathrm{Re}\,\Omega)^2 = (\mathrm{Im}\,\Omega)^2$

**Theorem [Athreya-Fan-L.]** Let $P$ be a positive-definite plane in $H^2(X,\mathbb{R})$.
Then

$$SL_P(R) \leq C \cdot R^{22} + o(R^{22})$$

where $C$ is a constant independent of the choice $P \subseteq H^2(X,\mathbb{R})$